Design of compound curves adapted to the satellite measurements

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Abstract: *The paper deals with a novelty related to the design of a region for an alternative railway track direction adequate for mobile satellite measurements. The new approach may become particularly useful in the design of the existing track axis control when the determination of both the main track directions becomes impossible. The only solution in that case is to apply to the geometrical system two circular arcs of a different radius, namely, to take advantage of compound curves. In the presented method the design of a new system is carried out by the use of the local system of coordinates. The solution of the design problem is aided by mathematical recording and is based on the determination of some universal equations describing the entire geometrical system. This procedure takes place sequentially involving consecutive parts of the system. The method has been illustrated by an appropriate calculation example.*

Key words: *railway track, satellite measurements, geometrical system, compound curves, design method*

1. Introduction

The global positioning system GPS (Specht, 2007) opens up possibilities for determining the coordinates of points in a uniform 3-D system of coordinates WGS 84 the outset of which is in the centre of the Earth's mass. Some improvements in railway track effective measurements have been made by a research team of the Gdansk University of Technology in cooperation with the Polish Naval Academy in Gdynia, relating to the technique of continuous (mobile) satellite measurements (Koc and Specht, 2009; Koc and Specht, 2010; Koc and Specht, 2011; Specht et al., 2011) by making inspection tours along railway track sections under test using antennas installed on a moving rail-vehicle.

Effective use of the GPS for the design, inventory and exploitation has become possible in Poland since the mid-2008 when the national Active Geodetic Network (ASG−EUPOS) was put into operation. As early as the beginning of 2009 a research team of the Gdańsk University of Technology and the Polish Naval Academy in Gdynia, in cooperation with the Institution of Polish Railway Lines PKP PLK S.A. in Gdynia, and also the Leica Geosystems Company carried out an experiment in field on the utilization of satellite mobile techniques to determine the position of the railway track axis. The measurements were of a pilot investigation and were aimed at finding out if the satellite method of continuous measurements can prove useful for the determination of the geometrical shape of track in horizontal plane.

By the use of a trailer (a flat car) PWM-15 and a rail tractor WM-15 it was possible to produce a prototype of the railway measuring set. The floor of the trailer is equipped with specially designed and constructed steel footings for installing on it some levelling heads needed for antennas to receive satellite signals. The first satellite measurements (carried out on a railway line measuring length of over 30 km) have already proved that the applied measuring technique opens up entirely new perspectives. Its utilization allows for very precise determination of data necessary for the design of railway track modernization (main directions and the turning angle of the route) (Koc et al., 2012; Koc et al., 2013). Making use of the obtained results one can verify the measuring technique, the applied equipment and its positioning.

The mobile satellite measurements make it possible to determine the coordinates of the existing railway track in the Polish state coordinate system PUW 2000 (Regulation of the Council of Ministers 2000, 2012) hereafter referred to as the system 2000. Under such circumstances it is obvious that the system should also be provided with the coordinates of the newly designed track axis necessary for the layout of the route in field. This calls for a change of the current design procedure.

General principles for designing track geometrical layout have been established in the early days of

railway development during the first half of the XIX century. Although the method of designing has a very long tradition it had been improved for the decades, the effects of this process have been shown in the fundamental work of Bałuch (1983). The developed calculation algorithms were then implemented in commercial Computer Aided Design software. However, searching for new solutions were continued and the issue of transition curves were an essential part of it (Arslan et al., 2014; Kobryń, 2014; Tari and Baykal, 2005; Tasci and Kuloglu, 2011; Zboiński and Woźnica, 2010). Characteristic feature of the used designing methodology is taking into account each element of the layout separately i.e. strait sections, circular arcs, transition curves. In the joints of this elements some problems appeared; the designing of the returning route various simplifying assumptions were performed. However, such geometrical layout should be treated as a one system with and calculated in an exact way. This conditions are satisfied for the analytical method of designing presented in this work. Examples of new methods for designing the region of the route direction change, accomodated to the mobile satellite measurements technique are given in papers (Koc, 2011; Koc, 2012a; Koc, 2012b). The new computer-aided design programs have also been prepared (Koc and Chrostowski, 2012; Koc and Chrostowski, 2014).

To make use of the obtained measuring data the region for the alternative route direction, being of interest to us, should be separated from the entire geometric system and subjected to an adequate transformation (rearrangement and rotation) of the coordinate system (Koc and Specht, 2009). The most advantageous situation will be if the new coordinate system of x, y can ensure a symmetrical position of the geometric system corresponding to the plotted main directions of the route (as seen in Fig. 1). In other words, the angles of both main routes directions with respect to the horizontal axis should be the same. Here the following relations should be satisfied (Korn GA and Korn TM, 1983):

$$x = (Y - Y_0) \cos \beta + (X - X_0) \sin \beta \qquad (1)$$
$$y = -(Y - Y_0) \sin \beta + (X - X_0) \cos \beta \qquad (2)$$

where:

Y − abscissa of the measuring point in the system 2000,

X − ordinate of the measuring point in the system 2000,

Y_0 − abscissa of the outset of the local system of coordinates,

X_0 − ordinate of the outset of the local system of coordinates,

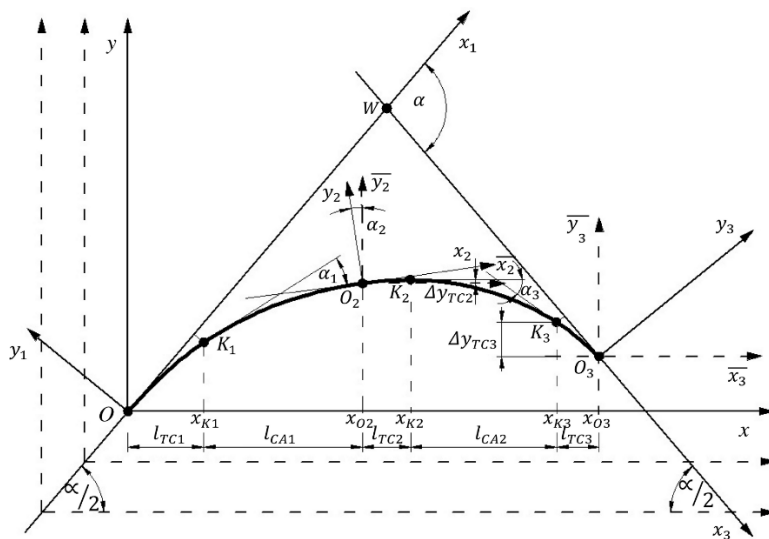β − turning angle of the system 2000.

Fig. 1. The whole geometric system analyzed in view of the local coordinate system

In the situation of a region to be subject to route direction change the design will most often be based on such a correction of the circular arc radius and type and lengths of the transition curves that the new geometric system can be most advantageous from the point of view of rail vehicle motion kinematics (i.e. the generation of transverse acceleration is as little as possible and changing smoothly), and that at the same time its position in the horizontal plane does not divert too much from the current one. As the satellite measurements, that have been carried out so far, indicate, the shape of the railway track in use is often so deformed that the determination of the main directions is impossible. Thus the use of a model system in the design based on the pattern transition curve − circular arc − transition curve, is excluded. The only problem to be solved is then to introduce into the geometrical system two circular arcs of different radii, which means to use compound curves.

According to the European Standard (EN 13803-1:2010) the compound curve "is sequence of curved alignment elements, including two or more circular curves in the same direction; the compound curve may include transition curves between the circular curves and / or the circular curves and the straight tracks". It is exactly this question that has become the object of this study. It should be pointed out at this place that the investigation deals here both with a symmetric (Koc, 2011; Koc, 2012a) (i.e. consisting of circular arc and the same type and length two transition curves) and also as an asymmetric (Koc, 2012b) (the type and length of the of transition curves may vary) approach to the problem. In fact the compoud curves can easily be reduced to a single circular arc with a determined radius.

In the presented method the design of a new geometric system takes place within the local coordinate system *x*, *y* (Fig. 1). Every geometric system can be provided with numerous local systems of coordinates (*LCS*), each of which is determined in terms of system 2000 by the adopted coordinates of its initial point. However, the position of the initial point for the newly designed geometric system at this stage is not known. It is only certain that the point should outline the beginning of the transition curve and lie on the straight which coincides with the main direction of the route. As soon as the whole design procedure with the system

x, *y* is completed it is possible to determine the coordinates of *LCS* initial point in the system 2000. The most significant element in the new procedure is the fact that the solution of the design problem is based on the determination of some universal equations describing the entire geometric system. Thus the creation of consecutive variants of the course of the route is not carried out by means of a graphic technique (e. g. by the use of AutoCAD program), but by the application of consecutive design values to appropriate mathematical formulate (radii of arcs and lengths of transition curves). An analytical recording is made sequentially covering successive parts of the geometric system of the route: the first transition curve (*TC1*), the first circular arc (*CA1*), the second transition curve (*TC2*), the second circular arc (*CA2*), and the third transition curve (*TC3*). The major accepted principle is to ensure conformity with tangents at connection points between particular geometrical elements.

2.   First Transition Curve (*TC1*)

The procedure begins with the determination of the ordinates of the transition curve *TC1* of length l_1, connecting the straight with the circular arc *CA1* of radius R_1, situated in the auxiliary system of coordinates Ox_1y_1 (Fig. 1). The choice of the type of curve defines the function of its curvature $k_1(l)$ on the basis of which it is possible to formulate the transition curve expressed in parametric form $x_1(l)$ and $y_1(l)$ (where parameter l is the position of a given point along the length of a curve).

In view of further proceeding the value of tangent $y_1'(l_1)$ is significant at the end of the curve occurring in the formula for this tangent at the end of curve *TC1* used in coordinate system Oxy.

$$y_1'(l_1) = -\tan\Theta_1(l_1)$$

where $\Theta_1(l)$ −function of the angle of the tangent to the curve *TC1*

$$\Theta_1(l) = \int k_1(l)\,dl$$

For example, the clothoid transition curve with a linear curvature is characterized by the following function

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$$k_1(l) = \frac{l}{R_1 l_1}$$

and the inclination angle of the tangent

$$\Theta_1(l) = \frac{l^2}{2 R_1 l_1}$$

At the end of the transition curve *TC1* the following expression is valid

$$\Theta_1(l_1) = \frac{l_1}{2 R_1}$$

As it turns out, the same value $\Theta_1(l_1)$ is also at the end of other forms of transition curves, therefore regardless of the type of the transition curve

$$y_1'(l_1) = -\tan\frac{l_1}{2 R_1}$$

The next step in the operation is the transformation of transition curve *TC1* to the assumed local coordinate system by turning its reference system through angle $\alpha/2$. As a result of the operation, parametric equations of the transition curve in terms of the local coordinate system are obtained (Korn GA and Korn TM, 1983):

$$x(l) = x_1(l)\cos\frac{\infty}{2} - y_1(l)\sin\frac{\infty}{2} \tag{3}$$

$$y(l) = x_1(l)\sin\frac{\infty}{2} + y_1(l)\cos\frac{\infty}{2} \tag{4}$$

Parameter occurring in equations (3) and (4) $l \in \langle 0, l_1 \rangle$ while the abscissa of transition curve $x \in \langle 0, x_{K1} \rangle$, where:

$$x_{K1} = l_{TC1} = x_1(l_1)\cos\frac{\infty}{2} - y_1(l_1)\sin\frac{\infty}{2} \tag{5}$$

The final ordinate of transition curve *TC1* is

$$y_{K1} = y(l_{TC1}) = x_1(l_1)\sin\frac{\infty}{2} + y_1(l_1)\cos\frac{\infty}{2} \tag{6}$$

The value of the slope of the tangent at the end of curve *TC1* is described by formula

$$s_{K1} = \tan\left(-\frac{l_1}{2 R_1} + \frac{\infty}{2}\right) \tag{7}$$

3. First Circular Arc (*CA1*)

Schematic diagram for the determination of the circular arc *CA1* equation is presented in Fig. 2. Every characteristic sections of the geometrical layout were presented. The tangents intersecting in the point M_1 that are passing through points K_1 and O_2 are determined in a conventional way with their inclinations, in sequence s_{K1} and s_{O2}.

In order to form the equation of the circular arc *CA1*, it is necessary to determine at first the position of its centre, i. e. point $S_1(x_{S1}, y_{S1})$, in terms of the coordinate system *Oxy* (Fig. 2). The circular arc must be tangent to the transition curve *TC1* at its end, that is, at point K_1, whereas the radius of circular arc *CA1* will be situated on the straight perpendicular to the tangent lying at point K_1. The mid-point circular arc coordinates are calculated from the system of equations with the following conditions: 1^0 point S_1 is on this straight line, 2^0 the distance between points S_1 and K_1 is equal to R_1. The coordinates of point S_1 are as follows:

$$x_{S1} = x_{K1} + \frac{s_{K1}}{\sqrt{1 + s_{K1}^2}} R_1 \tag{8}$$

$$y_{S1} = y_{K1} - \frac{1}{\sqrt{1 + s_{K1}^2}} R_1 \tag{9}$$

Now it is possible to take down the circular arc *CA1* equation.

$$y(x)_{CA1} = y_{S1} + \left[R_1^2 - (x_{S1} - x)^2\right]^{\frac{1}{2}}, \; x \in \langle x_{K1}, x_{O2}\rangle \tag{10}$$

To find the position of the circular arc *CA1* end, i. e. point $O_2(x_{O2}, y_{O2})$, one should assume the length of arc l_{R1}, and then determine the coordinates of point $M_1(x_{M1}, y_{M1})$ lying on the intersection of tangents to *CA1* drawn from both ends of the arc (Fig. 2). The coordinates of point M_1 are calculated from the system of equations with the following conditions: 1^0 point M_1 is on the tangent to *CA1* in the point K_1, 2^0 the distance between points K_1 and M_1 is equal to:

$$\overline{K_1 M_1} = R_1 \tan \frac{\propto_1}{2}$$

Calculated coordinates are:

$$x_{M1} = x_{K1} + \frac{\tan \dfrac{\propto_1}{2}}{\sqrt{1 + s_{K1}^2}} \, R_1 \qquad (11)$$

$$y_{M1} = y_{K1} + \frac{s_{K1} \tan \dfrac{\propto_1}{2}}{\sqrt{1 + s_{K1}^2}} \, R_1 \qquad (12)$$

where $\propto_1$ is the tangents' turning angle of arc *CA1*

whose value is $\propto_1 = \dfrac{l_{R1}}{R_1}$.

Point O_2 lies on the straight which is tangent to *CA1* passing through point M_1 , while the slope of s_{O2} of the tangent to *CA1* at point O_2 is

$$s_{O2} = \tan\left(\mathrm{atan}\, s_{K1} - \propto_1\right) \qquad (13)$$

Hence it is possible to determine the coordinates of point $O_2(x_{O2}, y_{O2})$.

$$x_{O2} = x_{M1} + \frac{\tan \dfrac{\propto_1}{2}}{\sqrt{1 + s_{O2}^2}} \, R_1 \qquad (14)$$

$$y_{O2} = y_{M1} + \frac{s_{O2} \tan \dfrac{\propto_1}{2}}{\sqrt{1 + s_{O2}^2}} \, R_1 \qquad (15)$$
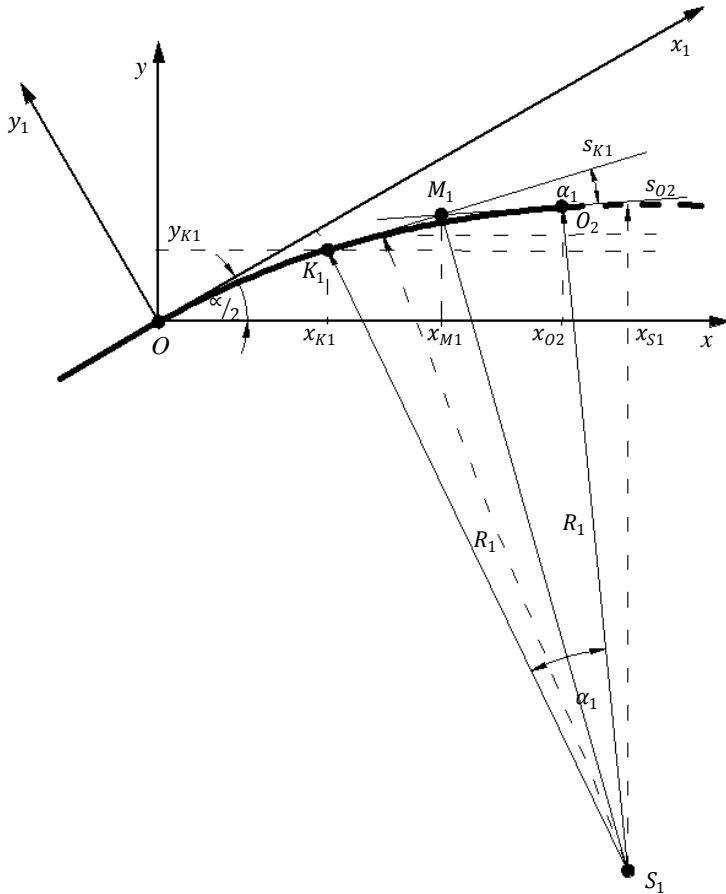

Fig. 2. Schematic diagram for the determination of the circular arc *CA1* equation

4. Second Transition Curve (*TC2*)

Transition curve *TC2* of length l_2 connects circular arcs of radii R_1 and R_2. It is found in the auxiliary system of coordinates $O_2x_2y_2$ (Fig. 1). The method of determining curvature $k_2(l)$ and its parametric equations $x_2(l)$ and $y_2(l)$, where $\langle l \in 0, l_2 \rangle$ has been presented in papers (Koc, 2014; Koc and Palikowska, 2012a; Koc and Palikowska, 2012b). In the case of a linear change of the curvature the obtained curve was referred to, in the above papers, as curve of class C^0, while with regard to nonlinear curvature (in the form of cubic equation) the curve was qualified as class C^1.

The value of tangent to *TC2* amounts to:

$$y_2'(l) = -\tan\left[\Theta_2(l)\right], \text{ where } \Theta_2(l) = \int k_2(l)\,dl \;.$$

At the end of *TC2* (i. e. for $l = l_2$)

$$y_2'(l_2) = -\tan\left[\Theta_2(l_2)\right]$$

In the case of linear curvature

$$y_2'(l_2) = -\tan\left[\frac{l_2}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right]$$

The transformation of *TC2*, into the local system of coordinates *x, y*, is carried out using the auxiliary system of coordinates $\overline{x_2}$, $\overline{y_2}$ (Fig. 1) obtained by a turn of the system $O_2x_2y_2$ through an angle of $\propto_2 = \text{atan}\, s_{O2}$.

The parametric equations of the transition curve *TC2* are as follows:

$$x(l) = x_{O2} + \overline{x_2}(l) \tag{16}$$

$$y(l) = y_{O2} + \overline{y_2}(l), \; \langle l \in 0, l_2 \rangle \tag{17}$$

whereas for $\propto_2 = \text{atan}\, s_{O2} > 0$

$$\overline{x_2}(l) = x_2(l)\cos\propto_2 - y_2(l)\sin\propto_2$$

$$\overline{y_2}(l) = x_2(l)\sin\propto_2 + y_2(l)\cos\propto_2, \; l \in \langle 0, l_2 \rangle$$

and for $\propto_2 = \text{atan}\, s_{O2} < 0$

$$\overline{x_2}(l) = x_2(l)\cos\propto_2 + y_2(l)\sin\propto_2$$

$$\overline{y_2}(l) = -x_2(l)\sin\propto_2 + y_2(l)\cos\propto_2, \; l \in \langle 0, l_2 \rangle$$

Hence the expressions for l_{TC2} and Δy_{TC2} are:

$$l_{TC2} = \left|\overline{x_2}(l_2)\right| \tag{18}$$

$$\Delta y_{TC2} = \left|\overline{y_2}(l_2)\right| \tag{19}$$

and the coordinates of point K_2 (of the end of *TC2*):

$$x_{K2} = x_{O2} + \overline{x_2}(l_2) \tag{20}$$

$$y_{K2} = y_{O2} + \overline{y_2}(l_2) \tag{21}$$

The value of tangent s_{K2} at point K_2 results from the inclination angle of the tangent at the end of the curve *TC2* in the auxiliary coordinate system x_2, y_2 (Fig. 1), performed in the coordinate system *x, y*.

$$s_{K2} = \tan\left[-\Theta_2(l_2) + \propto_2\right] \tag{22}$$

5. Second Circular Arc (*CA2*)

In Fig. 3 schematic diagram for the determination of the circular arc *CA2* equation is presented. Similarly, as in the case of the first circular arc, intersecting tangents passing through points K_2 and K_3 are determined in a conventional way with their inclinations, in sequence, s_{K2} and s_{K3}. To take down the equation for the circular arc *CA2*, it is necessary to determine the position of its centre, that is, point $S_2(x_{S2}, y_{S2})$ in the *Oxy* coordinate system. The circular arc should be tangent to transition curve *TC2* at its end, i. e. at point K_2, and the radius of the circular arc R_2 will be found on the straight perpendicular to the tangent lying at point K_2. At this stage of operation it is still too early to know the value of tangent s_{K3} at its end and the position of the arc end, namely, the coordinates of point $K_3(x_{K3}, y_{K3})$.

The coordinates of the circular arc *CA2* centre are as follows:

$$x_{S2} = x_{K2} + \frac{s_{K2}}{\sqrt{1 + s_{K2}^2}}\, R_2 \tag{23}$$
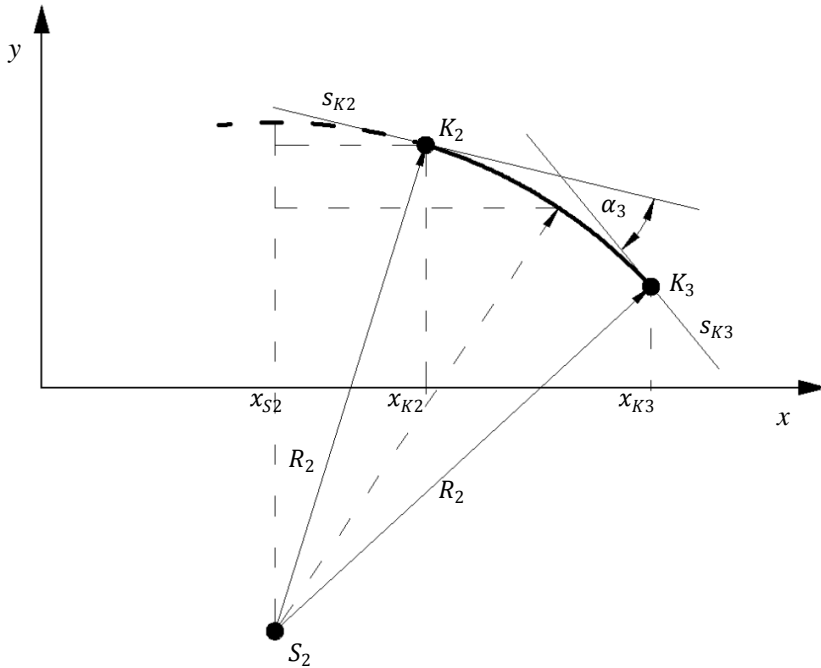
Fig. 3. Schematic diagram for the determination of the circular arc *CA2* equation

$$y_{S2} = y_{K2} - \frac{1}{\sqrt{1+s_{K2}^2}} R_2 \qquad (24)$$

The circular arc *CA2* can be also expressed in the form.

$$y(x)_{CA2} = y_{S2} + \left[R_2^2 - (x - x_{S2})^2 \right]^{\frac{1}{2}}$$
$$x \in \langle x_{K2}, x_{K3} \rangle \qquad (25)$$

6.   Third Transition Curve (*TC3*)

Transition curve *TC3* has length l_3 and is localized in the auxiliary system of coordinates $O_3 x_3 y_3$ (Fig. 4), but a precise position of point O_3 is still unknown at this stage (it is only obvious that it lies on the other direction of the main route). However, using this system it is possible to model the transition curve itself and to find out its basic data necessary for the determination of coordinates of point K_3, i. e. values l_{TC3} and Δy_{TC3}. The choice of the type of curve determines the function of its

curvature $k_3(l)$ on the basis of which one can form the parametric equations $x_3(l)$ and $y_3(l)$ of the transition curve.

In view of a further procedure of operation the value of tangent at the end of the curve, i. e. $y_3'(-l_3)$ becomes significant. It enables us to find out the value of this tangent within the local system of the *Oxy* coordinates. Regardless of the type of the transition curve

$$y_3'(-l_3) = \tan\left(\frac{l_3}{2 R_2} \right)$$

It is also possible to make a transformation of the curve points to the auxiliary system of coordinates $\overline{x_3}$, $\overline{y_3}$ (Fig. 4). For the reason that the axes of the system are parallel to the axis of the local *Oxy* system, the operation will be correct to determine l_{TC3} and Δy_{TC3}. The coordinates of the curve points in the $O_2 \overline{x_3} \overline{y_3}$ system are defined by parametric equations:

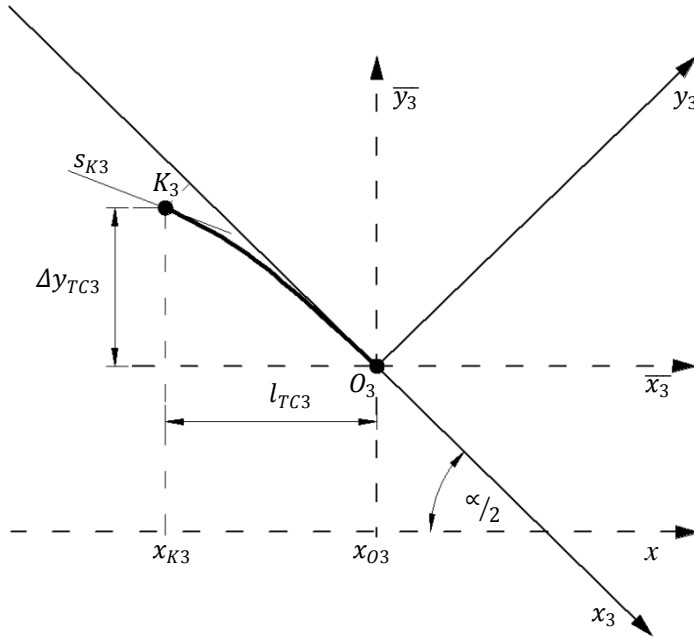Design of compound curves adapted to the satellite measurements

Fig. 4. Schematic diagram for the determination of the transition curve *TC3* equations

$$\overline{x}_3(l) = x_3(l)\cos\frac{\alpha}{2} + y_3(l)\sin\frac{\alpha}{2} \qquad (26)$$

$$\overline{y}_3(l) = -x_3(l)\sin\frac{\alpha}{2} + y_3(l)\cos\frac{\alpha}{2} \qquad (27)$$

Parameter $l \in \langle -l_3, 0 \rangle$, while the transition curve abscissa $x_3(l) \in \langle -l_{TC3}, 0 \rangle$, where

$$l_{TC3} = \left| \overline{x}_3(-l_3) \right| = \left| x_3(-l_3)\cos\frac{\alpha}{2} + y_3(-l_3)\sin\frac{\alpha}{2} \right| \quad (28)$$

The value of Δy_{TC3} is

$$\Delta y_{TC3} = \left| \overline{y}_3(-l_3) \right| = \left| -x_3(-l_3)\sin\frac{\alpha}{2} + y_3(-l_3)\cos\frac{\alpha}{2} \right| \quad (29)$$

One can also determine the value of tangent s_{K3} at point K_3 .

$$s_{K3} = \tan\left(\frac{l_3}{2R_2} - \frac{\alpha}{2} \right) \qquad (30)$$

7. Determination of Coordinates of the Missing Characteristic Points

At this stage of procedure aimed at a complete solution of the problem it is necessary to search for the coordinates of two characteristic points occurring in *LCS*: $K_3(x_{K3}, y_{K3})$ − of the end of circular arc *CA2* and transition curve *TC3*, and $O_3(x_{O3}, y_{O3})$ − the starting point of curve *TC3*. It is still too early to know the key value which present the coordinates of point $O(Y_o, X_o)$. − of the outset of *LCS* in terms of system 2000.

Starting from the equation of the tangent to *CA2*

$$y(x)'_{CA2} = -\frac{x - x_{S2}}{\left[R_2^2 - (x - x_{S2})^2 \right]^{\frac{1}{2}}}$$

and making use of condition $y(x_{K3})'_{CA2} = s_{K3}$, it is possible to determine x_{K3} .

$$x_{K3} = x_{S2} - \frac{s_{K3}}{\sqrt{1 + s_{K3}^2}} R_2 \qquad (31)$$

From equation (25) it follows that

$$y_{K3} = y_{S2} + \frac{1}{\sqrt{1 + s_{K3}^2}} R_2 \qquad (32)$$

Thus, the length of *CA2* projection upon axis *x* is

$$l_{CA2} = x_{K3} - x_{K2}$$

If the coordinates of point K_3 at the ends of *CA2* and *TC3* are known point $O_3(x_{O3}, y_{O3})$ can be found with ease.

$$x_{O3} = x_{K3} + l_{KP3} \qquad (33)$$
$$y_{O3} = y_{K3} - \Delta y_{KP3} \qquad (34)$$

Taking advantage of equations (26), (27), (33) and (34) one can obtain parametric equations for the transition curve *TC3* in *LCS*.

$$x(l) = x_{O3} + \overline{x_3}(l) = x_{O3} + x_3(l)\cos\frac{\infty}{2} + y_3(l)\sin\frac{\infty}{2} \quad (35)$$

$$y(l) = y_{O3} + \overline{y_3}(l) =$$
$$x_{O3} - x_3(l)\sin\frac{\infty}{2} + y_3(l)\cos\frac{\infty}{2}, \quad l \in \langle -l_3, 0 \rangle \qquad (36)$$

Knowing the coordinates of point $O_3(x_{O3}, y_{O3})$ it is not difficult to find the coordinates of point $W(x_W, y_W)$ – the intersection of the main directions (Fig. 1), and then the distance of that point from the outset of the local system of coordinates.

$$x_W = \frac{y_{O3} + \left(\tan\frac{\infty}{2}\right)x_{O3}}{2\left(\tan\frac{\infty}{2}\right)} \qquad (37)$$

$$y_W = \frac{1}{2}\left[y_{O3} + \left(\tan\frac{\infty}{2}\right)x_{O3}\right] \qquad (38)$$

$$\overline{OW} = \sqrt{x_W^2 + y_W^2} \qquad (39)$$

Owing to the above it is possible to determine the coordinates of the starting point of *LCS* in the system 2000.

$$Y_O = Y_W \pm \frac{\overline{OW}}{\sqrt{1 + B_1^2}} = \frac{A_2 - A_1}{B_1 - B_2} \pm \frac{\overline{OW}}{\sqrt{1 + B_1^2}} \qquad (40)$$

$$X_O = X_W \pm B_1 \frac{\overline{OW}}{\sqrt{1 + B_1^2}} =$$
$$A_1 + \frac{A_2 - A_1}{B_1 - B_2} B_1 \pm B_1 \frac{\overline{OW}}{\sqrt{1 + B_1^2}} \qquad (41)$$

where:

A_1, B_1 − parameters of the first main direction equation in the system 2000 $(X = A_1 + B_1 Y)$

A_2, B_2 − parameters of the second main direction equation in the system 2000 $(X = A_2 + B_2 Y)$

Y_W, Y_W − the coordinates of the intersection point of the main directions in the system 2000.

8.   Choice of the Design Variant

If the coordinates of point $O(Y_O, X_O)$ in terms of the system 2000 are known one can make a tnsformation of the values measured for *LCS* by the use of equations (1) and (2) for the purpose of comparing the existing geometric system with the new variant of the run of the route. For the assessment of the alternative design, advantage will be taken of index $\Delta y(x) = y(x) - y_{max}(x)$ describing certain variations in the ordinates.

If the obtained solution is not satisfactory (for the reason that, for example, values $\Delta y(x)$ are too big or their distribution along the length of the system is disadvantageous), some new values of geometric parameters are assumed and by the use of equations given in the paper another variant is generated. Having determined some coordinates of point $O(Y_O, X_O)$ adequate for the variant it can be compared with the measured geometric system.

On completing the design process, that is, the choice of the variant to work on, a transformation of it for the system 2000 is made with the aid of equations (Korn GA and Korn TM, 1983):

$$Y = Y_0 + x \cos\beta - y(x) \sin\beta \qquad (42)$$
$$X = X_0 + x \sin\beta + y(x) \cos\beta \qquad (43)$$

9.   Calculation Example

The presented algorithm used for the procedure will be illustrated by a calculation example relating to the

Design of compound curves adapted to the satellite measurements

design of the connection of two main directions of a railway line by the application of the geometrical elements given in Table 1. The turning angle of the route is 40^0, and the planned speed of trains is 110 km/h.

Table 1 is provided with a characteristic of the designed geometric system. There have been assumed values of radii for both the circular arcs and the length of arc *CA1*. Taking advantage of the kinematic conditions it was possible to find out the superelevation values along arcs and the transition curve lengths. The length of arc *CA2* is the resultant value and concludes the whole geometrical system. The values calculated for the characteristic points of the railway line (Fig. 1) are presented in Table 2.

The designed geometrical system is described by the following equations:

− transition curve *TC1* $x \in \langle 0, \; 75,471 \rangle$ m

$$x(l) = 0,939693\, x_1(l) - 0,34202\, y_1(l)$$
$$y(l) = 0,34202\, x_1(l) + 0,939693\, y_1(l)\;,\; l \in \langle 0, 80 \rangle \text{ m}$$

$$x_1(l) = l - \frac{1}{40 R_1^2 l_1^2} l^5 + \frac{1}{3456 R_1^4 l_1^4} l^9 - \frac{1}{599040 R_1^6 l_1^6} l^{13} + \ldots$$

$$y_1(l) = -\frac{1}{6 R_1 l_1} l^3 + \frac{1}{336 R_1^3 l_1^3} l^7 - \frac{1}{42240 R_1^5 l_1^5} l^{11} - \ldots$$

− circular arc *CA1* $x \in \langle 75,471, \; 220,593 \rangle$ m

$$y(x) = -1114,16 + \left[1200^2 - (448,086 - x)^2 \right]^{\frac{1}{2}}$$

− transition curve *TC2* $x \in \langle 220,593, \; 269,907 \rangle$ m

$$x(l) = 220,593 + \overline{x_2}(l)$$
$$y(l) = 64,079 + \overline{y_2}(l)\;,\; l \in \langle 0, 50 \rangle \text{ m}$$
$$\overline{x_2}(l) = x_2(l)\cos \alpha_2 - y_2(l)\sin \alpha_2$$
$$\overline{y_2}(l) = x_2(l)\sin \alpha_2 + y_2(l)\cos \alpha_2$$
$$\alpha_2 = 0,190733 \text{ rad}$$

$$x_2(l) = l - \frac{\left(\frac{1}{R_1}\right)^2}{6} l^3 - \frac{\frac{1}{R_1}}{8 l_2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) l^4 +$$

$$\left[\frac{\left(\frac{1}{R_1}\right)^4}{120} - \frac{1}{40\, l_2^2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^2 \right] l^5 +$$

$$\frac{\left(\frac{1}{R_1}\right)^3}{72\, l_2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) l^6 - \left[\frac{\left(\frac{1}{R_1}\right)^6}{5040} - \frac{\left(\frac{1}{R_1}\right)^2}{112\, l_2^2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^2 \right] l^7 -$$

$$\left[\frac{\left(\frac{1}{R_1}\right)^5}{1920\, l_2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) - \frac{\frac{1}{R_1}}{384\, l_2^3}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^3 \right] l^8 +$$

$$\left[\frac{\left(\frac{1}{R_1}\right)^8}{362880} - \frac{\left(\frac{1}{R_1}\right)^4}{1728\, l_2^2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^2 + \frac{1}{3456\, l_2^4}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^4 \right] l^9 + \ldots$$

Table 1. Characteristic of the designed geometrical system

Curve *TC1*	Arc *CA1*	Curve *TC2*	Arc *CA2*	Curve *TC3*
clothoid	$R_1 = 1200$ m	curve of class C^0	$R_2 = 700$ m	clothoid
$l_1 = 80$ m	$l_{R1} = 150$ m	$l_2 = 50$ m	$l_{R2} = 273,275$ m	$l_3 = 130$ m
	$h_1 = 70$ mm		$h_2 = 115$ mm	

Table 2. Comparison of numerical values for the characteristic points of the route

Point	O	K_1	O_2	K_2	K_3	O_3
Inclination s	0,36397	0,32666	0,19308	0,13500	− 0,26197	− 0,36397
Abscissa x [m]	0,000	75,471	220,593	269,907	540,946	664,376
Ordinate y [m]	0,000	26,523	64,079	72,288	55,730	15,085

$$y_2(l) = -\frac{\frac{1}{R_1}}{2}l^2 - \frac{1}{6l_2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)l^3 +$$

$$\frac{\left(\frac{1}{R_1}\right)^3}{24}l^4 + \frac{\left(\frac{1}{R_1}\right)^2}{20\,l_2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)l^5 - \left[\frac{\left(\frac{1}{R_1}\right)^5}{720} - \frac{\frac{1}{R_1}}{48\,l_2^2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^2\right]l^6 -$$

$$\left[\frac{\left(\frac{1}{R_1}\right)^4}{336\,l_2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) - \frac{1}{336\,l_2^3}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^3\right]l^7 +$$

$$\left[\frac{\left(\frac{1}{R_1}\right)^7}{40320} - \frac{\left(\frac{1}{R_1}\right)^3}{384\,l_2^2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^2\right]l^8 +$$

$$\left[\frac{\left(\frac{1}{R_1}\right)^6}{12960\,l_2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) - \frac{\left(\frac{1}{R_1}\right)^2}{864\,l_2^3}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^3\right]l^9 - \dots$$

− circular arc $CA2$ $x \in \langle 269,907, \;\; 540,946 \rangle$ m

$$y(x) = -621,42 + \left[700^2 - (x - 363,555)^2\right]^{\frac{1}{2}}$$

− transition curve $TC3$ $\langle x \in 540,946, \;\; 664,376 \rangle$ m

$$x(l) = 664,376 + 0,939693\,x_3(l) + 0,34202\,y_3(l)$$

$$y(l) = 15,085 - 0,34202\,x_3(l) + 0,939693\,y_3(l)$$

$$l \in \langle -130, 0 \rangle \text{ m}$$

$$x_3(l) = l - \frac{1}{40\,R_2^2\,l_3^2}l^5 + \frac{1}{3456\,R_2^4\,l_3^4}l^9 - \frac{1}{599040\,R_2^6\,l_3^6}l^{13} + \dots$$

$$y_3(l) = \frac{1}{6\,R_2\,l_3}l^3 - \frac{1}{336\,R_2^3\,l_3^3}l^7 + \frac{1}{42240\,R_2^5\,l_3^5}l^{11} - \dots$$

Fig. 5 presents the designed geometrical system using the rectangular coordinate system x, y. It is possible to note both main directions.
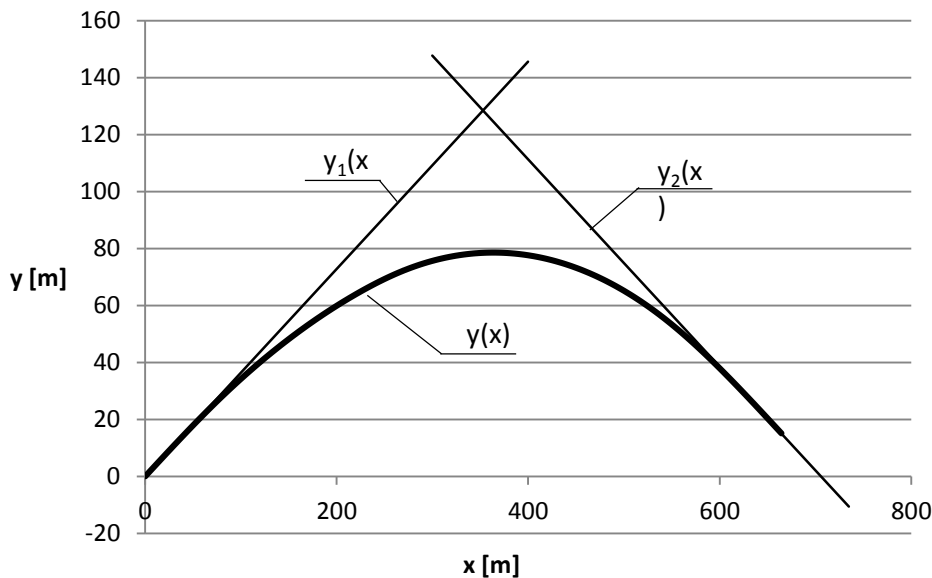

Fig. 5. Graphic illustration of the designed geometric system $y(x)$ (using a different scale); $y_1(x)$, $y_2(x)$ − main directions of the route

10. Summing-up

The application of mobile satellite measurements with the use the antennas installed on a mobile carriage travelling on rails, makes it possible to map the lay-out of the railtrack axis as far as the absolute frame of reference is concerned. This approach creates entirely new possibilities in the range of geometrical shaping of the rail tracks. Under conditions of the ensuring situation there arises the necessity to work out some new design methods.

In this paper a consecutive attempt has been made (following papers (Koc, 2011; Koc, 2012a; Koc, 2012b) to design the area related to the change of the railway line direction to comply with the continuous satellite measuring technique. This method may turn out become particularly useful in the design of axis control of the existing track when it is difficult to determine both the main directions of the railway route. The only solution in that situation is to apply two circular arcs of different radii to the geometrical system, in other words, to use compound curves.

The presented concepts of approach to the design of the region of the route direction change have led to an analytical solution by the use of some mathematical formulas, which are most friendly for practical applications. The design procedure is universal and makes it possible to diversify the type and length of the applied transition curves and the circular arcs. It should be treated as a generalized case of both the symmetric (Koc, 2011; Koc, 2012a) and the asymmetric (Koc, 2012b) solution of the problem for the reason that a compound curve can easily be reduced to a single circular arc of a determined radius.

The effects that follow from the application of the design method under consideration have been illustrated by a concrete calculation example. In order to put the given procedure in practice it will be necessary to prepare in the near future some appropriate computer-aided design method.

References

[1] ARSLAN, A., TARI, E., ZIATDINOV, R. and NABIYEV, R.I., 2014. Transition curve modeling with kinematical properties: research on log-aesthetic curves. *Computer-Aided Design and Applications,* 11(5), pp. 509-517.

[2] BAŁUCH, H., 1983. *Optymalizacja układów geometrycznych toru.* Warszawa: Wydawnictwa Komunikacji i Łączności.

[3] EN 13803-1:2010, June 2010. *Railway applications −Track −Track alignment design parameters −Track gauge 1435 mm and wider − Part 1: Plain line..* European Standard edn. European Committee for Standardization.

[4] KOBRYŃ, A., 2014. New solutions for general transition curves. *Journal of Surveying Engineering,* 140(1), pp. 12-21.

[5] KOC, W. and CHROSTOWSKI, P., 2012. Program komputerowy do projektowania zmiany kierunku trasy kolejowej. *Zeszyty NaukowoTechniczne Stowarzyszenia Inżynierów i Techników Komunikacji w Krakowie. Seria: Materiały Konferencyjne,* 3(99), pp. 213-223.

[6] KOC, W. and CHROSTOWSKI, P., 2014. Computer-aided design of railroad horizontal arc areas in adapting to satellite measurements. *Journal of Transportation Engineering,* 140(3), pp. 04013017 (1-8).

[7] KOC, W. and PALIKOWSKA, K., 2012a. Analiza sposobów modelowania krzywizny− krzywe Bezier a metoda analityczna. *Zeszyty Naukowo-Techniczne Stowarzyszenia Inżynierów i Techników Komunikacji w Krakowie. Seria: Materiały Konferencyjne,* 3(99), pp. 225-237.

[8] KOC, W. and PALIKOWSKA, K., 2012b. Ocena dynamiczna wybranych sposobów łączenia elementów trasy o zróżnicowanej krzywiźnie. *Technika Transportu Szynowego [CD-ROM],* 18(9), pp. 1785-1807.

[9] KOC, W. and SPECHT, C., 2009. Wyniki pomiarów satelitarnych toru kolejowego. *TTS Technika Transportu Szynowego,* 15(7-8), pp. 58-64.

[10] KOC, W. and SPECHT, C., 2010. Application of the Polish active GNSS geodetic network for surveying and design of the railroad, S. LAKUŠIĆ, ed. In: *First International Conference on Road and Rail Infrastructure : CETRA 2010, Opatija, Croatia,* 17-18 May 2010, Departament of Transportation Faculty of Civil Engineering University of Zagrzeb, pp. 757-762.

[11] KOC, W. and SPECHT, C., 2011. Selected problems of determining the course of railway routes by use of GPS network solution. *Archives of Transport,* 23(3), pp. 303-320.

[12] KOC, W., 2011. Metoda projektowania rejonu zmiany kierunku trasy kolejowej. *Problemy Kolejnictwa,* 152, pp. 197-217.

[13] KOC, W., 2012a. Design of rail-track geometric systems by satellite measurement. *Journal of Transportation Engineering,* 138(1), pp. 114-122.

[14] KOC, W., 2012b. Projektowanie rejonu zmiany kierunku trasy kolejowej w zapisie matematycznym. *Przegląd Komunikacyjny,* 67(7-8), pp. 96-101.

[15] KOC, W., 2014. Analytical method of modelling the geometric system of communication route. *Mathematical Problems in Engineering,* 2014, ID 679817.

[16] KOC, W., CHROSTOWSKI, P. and SPECHT, C., 2013. The application effects of continuous satellite measurements of railway lines, *Proceedings of the 12th International Conference and Exhibition–Railway Engineering London, UK,* 10-11 July 2013.

[17] KOC, W., SPECHT, C., CHROSTOWSKI, P. and PALIKOWSKA, K., 2012. The accuracy assessment of determining the axis of railway track basing on the satellite surveying. *Archives of Transport,* 24(3), pp. 307-320.

[18] KORN, G., A and KORN, T., M, 1983. *Matematyka dla pracowników naukowych i inżynierów.* Warszawa: Państwowe Wydawnictwo Naukowe.

[19] REGULATION OF THE COUNCIL OF MINISTERS of 15 November 2012. The regulation on the national spatial reference system. Journal of Laws of the Republic of Poland of 14 October 2012, item 1085. [ROZPORZĄDZENIE RADY MINISTRÓW Z DNIA 15 PAŹDZIERNIKA, 2012. *Rozporządzenie w sprawie państwowego systemu odniesień przestrzennych.* Dziennik Ustaw z dnia 14 listopada 2012 r., poz. 1247.]

[20] REGULATION OF THE COUNCIL OF MINISTERS of 8 August 2000. The regulation on the national spatial reference system. Journal of Laws of the Republic of Poland, no.70, item 1085. [ROZPORZĄDZENIE RADY MINISTRÓW Z DNIA 8 SIERPNIA, 2000. *Rozporządzenie w sprawie państwowego systemu odniesień przestrzennych.* Dziennik Ustaw nr 70, poz. 821].

[21] SPECHT, C., 2007. *System GPS.* Pelplin: BERNARDINUM.

[22] SPECHT, C., NOWAK, A., KOC, W. and JURKOWSKA, A., 2011. Application of the Polish Active Geodetic Network for Railway Track Determination. In: A. WEINTRIT and T. NEUMANN, eds, *Transport Systems and Processes: Marine Navigation and Safety of Sea Transportation.* London: CRC Press, pp. 77-82.

[23] TARI, E. and BAYKAL, O., 2005. A new transition curve with enhanced properties. *Canadian Journal of Civil Engineering,* 32(5), pp. 913-923.

[24] TASCI, L. and KULOGLU, N., 2011. Investigation of a new transition curve. *Baltic journal of road and bridge engineering,* 6(1), pp. 23-29.

[25] ZBOIŃSKI, K. and WOŹNICA, P., 2010. Optimisation of the railway transition curves' shape with use of vehicle-track dynamical model. *Archives of Transport,* 22(3), pp. 387-407.