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A GENERAL ITERATIVE SOLVER FOR UNBALANCED INCONSISTENT TRANSPORTATION PROBLEMS

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Abstract: *The transportation problem, as a particular case of a linear programme, has probably the highest relative frequency with which appears in applications. At least in its classical formulation, it involves demands and supplies. When, for practical reasons, the total demand cannot satisfy the total supply, the problem becomes unbalanced and inconsistent, and must be reformulated as e.g. finding a least squares solution of an inconsistent system of linear inequalities. A general iterative solver for this class of problems has been proposed by S. P. Han in his 1980 original paper. The drawback of Han's algorithm consists in the fact that it uses in each iteration the computation of the Moore-Penrose pseudoinverse numerical solution of a subsystem of the initial one, which for bigger dimensions can cause serious computational troubles. In order to overcome these difficulties we propose in this paper a general projection-based minimal norm solution approximant to be used within Han-type algorithms for approximating least squares solutions of inconsistent systems of linear inequalities. Numerical experiments and comparisons on some inconsistent transport model problems are presented.*

Key words: *inconsistent linear inequalities, least squares solutions, projection-type algorithms, Kaczmarz Extended, transportation problem, Simplex algorithm*

1. Introduction

The classical transportation problem involves sources $(S_i)_{i\in\{1,\ldots,n\}}$, where supplies $(s_i)_i = 1,\ldots,n$ of some goods are available, and destinations $(D_j)_{j\in\{1,\ldots,m\}}$, where some demands $(d_j)_{j=1,\ldots,m}$ are requested. The costs of shipping $(c_{ij})_{i\in\{1,\ldots,n\},j\in\{1,\ldots,m\}}$ for the transportation of one unit from source S_i to destination D_j become the entries of the $C: n\times m$ cost matrix (see Table 1).

If we denote by $x_{ij}, i=1,\ldots,n, j=1,\ldots,m$ the number of units transported from source S_i to destination D_j, we get the following mathematical model of the (classical) transportation problem:

$$\min \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij} \tag{1.1}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_{ij} \geq d_j,\ j=1,\ldots,m \tag{*}$$

$$\sum_{j=1}^{m} x_{ij} = s_i,\ i=1,\ldots,n \tag{**}$$

$$x_{ij} \geq 0, i=1,\ldots,n, j=1,\ldots m$$

Table 1: The classical transportation problem

	D_1	D_2	D_3	$\ldots$	D_m	Supply(s)
S_1	c_{11}	c_{12}	c_{13}	$\ldots$	c_{1m}	s_1
S_2	c_{21}	c_{22}	c_{23}	$\ldots$	c_{2m}	s_2
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
S_n	c_{n1}	c_{n2}	c_{n3}	$\ldots$	c_{nm}	s_n
Demand(d)	d_1	d_2	d_3	$\ldots$	d_m	

The problem is called **balanced** if the total supply equals the total demand (i.e. $\sum_{i=1}^{n} s_i = \sum_{j=1}^{m} d_j$), and **unbalanced** otherwise. In the balanced case or the unbalanced one with $\sum_{i=1}^{n} s_i \geq \sum_{j=1}^{m} d_j$, the linear program (1.1) is consistent and well known methods (including Simplex-type algorithms) are available (see Koopmans and Beckmann, 1957; Popa et al., 2013). We will consider in this paper the unbalanced case

$$\sum_{i=1}^{n} s_i < \sum_{j=1}^{m} d_j \tag{1.2}$$

for which the linear program (1.1) becomes inconsistent (i.e. the set of feasible solutions is empty).

In the paper (Han, 1980) the author showed that the problem (1.1) can be written as an inconsistent system of linear inequalities

$$Ax \leq b \tag{1.3}$$

In Han (1980), (see also Carp et al., 2015), Han presented an iterative method for solving (1.3). For this, he transformed (1.3) in an optimization problem of the form: find $x \in \Re^n$ such that

$$f(x) = \min\{f(z), z \in \Re^n\} \quad \text{where} \quad f(z) = \frac{1}{2} \| (Az - b)_+ \|^2 \tag{1.4}$$

and for $y \in \Re^m, y_+ \in \Re^m$ is defined by

$$(y_+)_i = \max\{y_i, 0\} \tag{1.5}$$

If (1.3) is inconsistent, then for any $x \in \Re^n$ the set $I(x) = \{i, \langle A_i, x\rangle \geq b_i\}$ is non-empty. If $I(x) = \{i_1, \ldots, i_p\}$ we will suppose that $i_1 < i_2 < \ldots < i_p$; $A_{I(x)}, b_{I(x)}$ will denote the submatrix of A with the rows $A_{i_1}, \ldots, A_{i_p}$ and the subvector of b with components $b_{i_1}, \ldots, b_{i_p}$.

With these notations, we present below Han's algorithm (**H**), firstly proposed by Han (1980) (B^+ will denote the Moore-Penrose pseudoinverse of B).

Algorithm H *Let* $x^0 \in \Re^n$ *be arbitrary fixed; for* $k = 0,1,\ldots$ *do:*

Step 1. *Find* $I_k = I(x^k)$ *and compute* $d^k \in \Re^n$ *as the (unique) minimal norm solution of the linear least squares problem*

$$\| A_{I_k} d - (b_{I_k} - A_{I_k} x^k) \| = \\ = \min!, \text{i.e. } d^k = A_{I_k}^+ (b_{I_k} - A_{I_k} x^k) \tag{1.6}$$

Step 2. *Compute* $\lambda^k \in \Re$ *as the smallest minimizer of the function*

$$\theta(\lambda) = f(x^k + \lambda d^k), \lambda \in \Re. \tag{1.7}$$

Step 3. *Set* $x^{k+1} = x^k + \lambda^k d^k$.

The existence of the smaller minimizer for the convex function θ from (1.7) was explained in Han (1980) and an algorithmic procedure to find it was given in Popa et al. (2013). The following result was proved in Han (1980).

Theorem 1 *Let* $(x^k)_{k\geq 0}$ *be the sequence generated by the algorithm* **H**.

- *Either it exists an integer* $k_0 \geq 0$ *such that* $\lim_{k\to\infty} f'(x^k) = 0$.
- *If* $z^k = (Ax^k - b)_+$ *then, for some* x^* *solution of (1.4),* $\lim_{k\to\infty} z^k = z^* = (Ax^* - b)_+$.
- *The algorithm **H** produces from any starting point* x^0*, a solution for (1.4), in a finite number of iterations (in exact arithmetics).*

For the approximate computation of the minimal norm solution from **Step 1** (1.6), we proposed in Carp et al. (2015) the Kaczmarz Extended (**KE**) algorithm from Popa (1998). The obtained algorithm is the **HKE** below.

Algorithm HKE *Let* $x^0 \in \Re^n$ *be arbitrary fixed; for* $k = 0,1,\ldots$ do*:*

Step 1. *Find* $I_k = I(x^k)$ *and compute an approximation* $d^{k,j} \in \Re^n$ *of the minimal norm solution* d^k *from (1.6), by performing* $j \geq 1$ *iterations of the algorithm **KE**.*
Step 2. *Compute* $\lambda^{k,j} \in \Re$ *as the smallest minimizer of* $\theta(\lambda) = f(x^k + \lambda d^{k,j}), \lambda \in \Re$
Step 3. *Set* $x^{k+1} = x^k + \lambda^{k,j} d^{k,j}$.

Theorem 2 *Let us suppose that there exist an integer* $J \geq 1$, *and constants* $C_1 \in [0,1), C_2 \geq 0$, *all independent on the iteration index* k *, such that*

$$\begin{aligned} &\| A_{I_k}(d^{k,J} - d^k) \| \leq C_1 \| A_{I_k} d^{k,J} \| \\ &\| d^{k,J} - d^k \| \leq C_2 \| A_{I_k} d^{k,J} \| \end{aligned} \quad (1.8)$$

$\forall k \geq 0$, *where* d^k *is the minimal norm solution of (1.6) and* $d^{k,J}$ *the approximation generated after* J *iterations of **KE** in Step 1 of **HKE**. Then (i) and (ii) from Theorem 1 hold for the sequence* $(x^k)_{k \geq 0}$ *generated by the algorithm **HKE**.*

The replacement of the original Han's pseudoinverse solver (1.6) with the iterative approximation given by the projection-based **KE** algorithm from Popa (1998) has already been analyzed in Carp et al. (2015). But, the **KE** algorithm is not always oriented to the sparsity structure of the system matrix A from (1.3), which may slow down the convergence speed. In the present paper we propose the replacement of the iterative solver **KE** from Step 1 of the algorithm **HKE** with a sparsity oriented method, algorithm **DWE** from Popa (2010). For this, we firstly introduce in Step 1 a more general projection-type method (denoted by **ALG**). In section 2 we prove a result as in Theorem 2 above for the Han-type algorithm obtained, whereas in section 3 we present some numerical experiments and comparisons on some inconsistent transport model problems, involving as iterative solvers the algorithms **HKE** and **HDWE**.

2. Algorithm Han with a general projection-based minimal norm solution approximant

We will consider the general linear least squares problem: find $x \in \Re^n$ such that:

$$\| Bx - c \| = \min\{ \| Bz - c \|, z \in \Re^n \} \quad (2.1)$$

where $\| \cdot \|$ denotes the Euclidean norm ($\langle \cdot, \cdot \rangle$ will be the Euclidean scalar product). Concerning the matrix involved in (2.1) we will suppose for the whole paper that it has nonzero rows B_i and columns B^j, i.e.

$$B_i \neq 0, i = 1, \ldots, m, B^j \neq 0, j = 1, \ldots, n \quad (2.2)$$

These assumptions are not essential restrictions of the generality of the problem (2.1) because, if B has null rows and/or columns, it can be easily proved that they can be eliminated without affecting its set of classical or least squares solutions. We first introduce some notations: the spectrum and spectral radius of a square matrix will be denoted by $\sigma(B)$ and $\rho(B)$, respectively. By B^T, $N(B), R(B)$ we will denote the transpose, null space and range of B. $P_S(x)$ will be the orthogonal (Euclidean) projection onto a vector subspace S of some $\Re^q$. $S(B;c), LSS(B;c), x_{LS}$ will stand for the set of classical and least squares solutions of (2.1), respectively, and the (unique) minimal norm one. In the consistent case for (2.1) we have $S(B;c) = LSS(B;c)$. In the general case the following properties are known

$$\begin{aligned} &x_{LS} \perp N(B), c = c_B + c_B^* \\ &\textit{with } c_B = P_{R(B)}(c), c_B^* = P_{N(B^T)}(c) \end{aligned} \quad (2.3)$$

$$\begin{aligned} &LSS(B;c) = x_{LS} + N(B) \\ &\textit{and } x \in LSS(B;c) \Leftrightarrow Bx = c_B \end{aligned} \quad (2.4)$$

$$\begin{aligned} &S(B;c) = x_{LS} + N(B) \\ &\textit{and } x \in S(B;c) \Leftrightarrow Bx = c \end{aligned} \quad (2.5)$$

Let $Q: n \times n$ and $R: n \times m$ be real matrices satisfying the (main) assumptions

$$I - Q = RB \quad (2.6)$$

$$\textit{if } \tilde{Q} = QP_{R(B^T)}, \textit{ then } \| \tilde{Q} \| < 1 \quad (2.7)$$

where $\| \tilde{Q} \|$ denotes the spectral norm of the matrix $\tilde{Q}$. In the paper Nicola et al. (2011) we proposed the following algorithm.

Algorithm General Projections with Correction (GPC)

Initialization: $x^0 \in \Re^n$

Iterative step:

$$x^{k+1} = Qx^k + Rc + v^k \tag{2.8}$$

and we introduced the additional assumptions:

$$\forall y \in \Re^m, Ry \in \mathrm{R}(B^T) \tag{2.9}$$

$$v^k \in R(B^T), \forall k \geq 0 \tag{2.10}$$

and for γ^k defined by

$$\gamma^k = v^k + Rc_B^* \tag{2.11}$$

we supposed that there exist constants $C > 0$ and $\delta \in [0,1)$ such that

$$\| \gamma^k \| \leq c\delta^k, \forall k \geq 0 \tag{2.12}$$

Then, the algorithm **HGPC** is exactly **HKE**, but with the **KE** solver in Step 1 replaced by the more general **GPC** one. We will rewrite the least squares problem (1.6) as

$$\| Bd - c \| = min! \tag{2.13}$$

Then the inequalities (1.8) become

$$\| B\varepsilon^J \| \leq C_1 \| Bd^J \|, \| \varepsilon^J \| \leq C_2 \| Bd^J \| \tag{2.14}$$

where d_{LS} is the minimal norm solution of (2.13), d^J the approximation generated after $J \geq 1$ steps of the algorithm **GPC**, and $\varepsilon^J = d^J - d_{LS}$ is the corresponding error.

Lemma 1 *If*

$$\| B\varepsilon^J \| \leq \frac{1}{4} \| Bd_{LS} \|, \| \varepsilon^J \| \leq \frac{1}{4} \| Bd_{LS} \| \tag{2.15}$$

then (2.14) holds with $C_1 = C_2 = \frac{1}{2}$

Proof. Indeed, by using the first inequality in (2.15) we get

$$\| Bd^J \| \geq$$

$$\geq \| Bd_{LS} \| - \| B\varepsilon^J \| \geq \| Bd_{LS} \| - \frac{1}{4} \| Bd_{LS} \| > \frac{1}{2} \| Bd_{LS} \|$$

If we reverse it and use again the first inequality in (2.15) we get

$$\frac{1}{\| Bd^J \|} \leq \frac{2}{\| Bd_{LS} \|} \Rightarrow \frac{\| B\varepsilon^J \|}{\| Bd^J \|} \leq \frac{2 \| B\varepsilon^J \|}{\| Bd_{LS} \|} \Rightarrow$$

$$\frac{\| B\varepsilon^J \|}{\| Bd^J \|} \leq \frac{2\frac{1}{4} \| Bd_{LS} \|}{\| Bd_{LS} \|} = \frac{1}{2}$$

i.e. the first inequality in (2.14). Finally, from the first inequality in (2.15) we get the first inequality in (2.14), which combined with the second inequality in (2.15) give us

$$\| \varepsilon^J \| \leq \frac{1}{4} \| Bd_{LS} \| \leq \frac{1}{4} 2 \| Bd^J \| \leq \frac{1}{2} \| Bd^J \|$$

i.e the second inequality in (2.14).

Theorem 3 *The conclusions of Theorem 2 hold also for the* **HGPC** *algorithm.*

Proof. It suffices to prove that it exists an iteration $J \geq 1$ such that the inequalities (2.15) hold. Because $d^0 = 0$, we have, from (2.4),(2.6) and (2.11)

$$\varepsilon^0 = -d_{LS}, \ \varepsilon^{j+1} = Q\varepsilon^j + \gamma^j, j \geq 0 \tag{2.16}$$

Because $Q\varepsilon^0 = -Qd_{LS} = -\tilde{Q}d_{LS} = \tilde{Q}\varepsilon^0$, $Q\tilde{Q} = (P_{\mathrm{N}(B)} + \tilde{Q})\tilde{Q} = \tilde{Q}\tilde{Q}$ and for $\gamma^k \in \mathrm{R}(B^T)$, $Q\gamma^k = (P_{\mathrm{N}(B)} + \tilde{Q})\gamma^k = \tilde{Q}\gamma^k$, from (2.8) we get

$$\varepsilon^{j+1} = \tilde{Q}\varepsilon^j + \gamma^j, j \geq 0 \tag{2.17}$$

and, by a recursive argument,

$$\varepsilon^j = \tilde{Q}\varepsilon^0 + \sum_{i=0}^{j-1}\tilde{Q}^{j-1-i}\gamma^i, \forall j \geq 1 \quad (2.18)$$

Thus, by taking norms and using (2.16), we get

$$\|\varepsilon^j\| \leq \|\tilde{Q}\|^j \|d_{LS}\| + j\mu^j \frac{C}{\|d_{LS}\|}\|d_{LS}\| \leq \leq (\mu^j + j\mu^j \hat{C})\|d_{LS}\| \quad (2.19)$$

where

$$\hat{C} = \frac{C}{\|d_{LS}\|}, \mu = max\{\|\tilde{Q}\|, \delta\} < 1 \quad (2.20)$$

Then, by multiplying in (2.18) with B and taking norms, we get

$$\|B\varepsilon^j\| \leq \leq \|B\|_2 \|\tilde{Q}\|^j \|d_{LS}\| + \|B\|_2 \; j\mu^j \hat{C}\|d_{LS}\| \leq \leq (\mu^j + j\mu^j \hat{C})\|B\|_2 \|d_{LS}\| \quad (2.21)$$

Hence, (2.15) will hold if

$$\mu^j + j\mu^j\beta \leq \frac{\|Bd_{LS}\|}{4\|B\|_2\|d_{LS}\|} \leq \frac{1}{4}$$

and $\mu^j + j\mu^j\beta \leq \frac{\|Bd_{LS}\|}{4\|d_{LS}\|} \leq \frac{1}{4}\|B\|_2$

i.e. if we have

$$\mu^J + J\mu^J\beta \leq \frac{1}{4}\min\{1, \|B\|_2\} \quad (2.22)$$

Because $\lim_{j\to\infty}(\mu^j + j\mu^j\beta) = 0$ it exists a smallest integer $J \geq 1$ for which (2.22) holds, $\forall j \geq J$ and the proof is complete.

Remark 1 *The general method* **GPC** *covers many standard projection based algorithms: Kaczmarz Extended, Cimmino Extended, Projection Jacobi Extended, DW Extended, etc. (see Popa, 2012). The advantage of this consists in the case the system matrix B is sparse more efficient algorithms as DW Extended from Popa (2010) can be used (see also our Numerical Experiments section).*

3. Numerical experiments

We will consider in our numerical experiments an unbalanced and inconsistent transport problem **P** described by (1.1) and Table 2, where we suppose that there exist 7 sources, $S_i, i = 1,\ldots,7$, 7 destinations, $D_j, j = 1,\ldots,7$, and (x_{ij}) represents quantities of merchandise shipped between source S_i and destination D_j. The transportation problem **P** being inconsistent (because $\sum_{i=1}^{7}s_i = 4000 < 4145 = \sum_{i=1}^{7}d_i$), so will be the system of linear inequalities equivalent with solving it in a least squares sense $Ax \leq b$, which motivates us to solve it with Han type algorithms. In order to have a comparative analysis, we also applied the Simplex algorithm on the problem.

Table 2: The unbalanced container transportation problem

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	**Supply(s)**
S_1	3	3	4	12	20	5	9	**1050**
S_2	7	1	5	3	6	8	4	**350**
S_3	5	4	7	6	5	12	3	**470**
S_4	4	5	14	10	9	8	7	**600**
S_5	8	2	12	9	8	4	2	**600**
S_6	6	1	8	7	2	3	1	**480**
S_7	9	10	6	8	7	6	5	**450**
Demand(d)	**455**	**320**	**540**	**460**	**760**	**830**	**780**	

All the computations were made in Matlab R2010a, a widely used software for mathematical, science and engineering applications. We made use of the build-in Matlab implementation of Simplex algorithm, *linprog*, whereas the Han-type algorithm was programmed as user-defined function.

All runs are started with the initial approximations $x_0 = (y_0^T, 0)^T$, with $y_0 \geq 0$, and are terminated if at the current iteration x^k satisfy:

$$\|A^T(Ax^k - b)_+\| \leq 10^{-14} \quad (3.1)$$

The problem **P** has the restrictions $(1.1:(*)-(**))$ corresponding to the relations (3.2).

$$\begin{aligned} &x_{11}+x_{21}+x_{31}+x_{41}+x_{51}+x_{61}+x_{71}\geq 455\\ &x_{12}+x_{22}+x_{32}+x_{42}+x_{52}+x_{62}+x_{72}\geq 320\\ &x_{13}+x_{23}+x_{33}+x_{43}+x_{53}+x_{63}+x_{73}\geq 540\\ &x_{14}+x_{24}+x_{34}+x_{44}+x_{54}+x_{64}+x_{74}\geq 460\\ &x_{15}+x_{25}+x_{35}+x_{45}+x_{55}+x_{65}+x_{75}\geq 760\\ &x_{16}+x_{26}+x_{36}+x_{46}+x_{56}+x_{66}+x_{76}\geq 830\\ &x_{17}+x_{27}+x_{37}+x_{47}+x_{57}+x_{67}+x_{77}\geq 780\\ \\ &x_{11}+x_{12}+x_{13}+x_{14}+x_{15}+x_{16}+x_{17}=1050\\ &x_{21}+x_{22}+x_{23}+x_{24}+x_{25}+x_{26}+x_{27}=350\\ &x_{31}+x_{32}+x_{33}+x_{34}+x_{35}+x_{36}+x_{37}=470\\ &x_{41}+x_{42}+x_{43}+x_{44}+x_{45}+x_{46}+x_{47}=600\\ &x_{51}+x_{52}+x_{53}+x_{54}+x_{55}+x_{56}+x_{57}=600\\ &x_{61}+x_{62}+x_{63}+x_{64}+x_{65}+x_{66}+x_{67}=480\\ &x_{71}+x_{72}+x_{73}+x_{74}+x_{75}+x_{76}+x_{77}=450 \end{aligned} \tag{3.2}$$

First, we renumber the unknowns as

$$x_{ij}\rightarrow y_l, i\in\{1,\ldots 7\}, j\in\{1,\ldots 7\}, l\in\{1,\ldots 49\} \tag{3.3}$$

Let $c\in\Re^{49}$ be the cost vector of the problem **P** and B_1, B_2 the 7×49 matrices corresponding to the first 7 inequalities, respectively the last 7 equalities in (3.2). Also, let $d, s\in\Re^7$ be defined by (see the right hand sides in (3.2)):

$$\begin{aligned} d &= (455,320,540,460,760,830,780)^T\\ s &= (1050,350,470,600,600,480,450)^T \end{aligned} \tag{3.4}$$

Then, the above problem **P** can be written as follows:

$$\min\langle c,y\rangle \text{ s.}t.\ B_1 y\geq d,\ B_2 y = s,\ y\geq 0 \tag{3.5}$$

If we define the matrix $B: 21\times 49$ by (in Matlab notation)

$$B=[B_1^T; B_2^T; -B_2^T]^T \tag{3.6}$$

and the vector $\hat{d}\in\Re^{21}$ by

$$\hat{d}=[d^T; s^T; -s^T]^T \tag{3.7}$$

the problem **P** can be written as

$$\min\langle c,y\rangle \text{ s.}t.\ By\geq \hat{d},\ y\geq 0 \tag{3.8}$$

with the corresponding dual problem given by (see e.g. Vanderbei, 2001)

$$\max\langle \hat{d},u\rangle \text{ s.}t.\ B^T u\leq c,\ u\geq 0 \tag{3.9}$$

respectively. Solving the pair of dual problems (3.8) and (3.9) is equivalent with solving the system of inequalities $Ax\leq b$, where $x=[y^T;u^T]^T\in\Re^{49}\times\Re^{21}$ and A and b are constructed as in (13) from Carp et al. (2010).

Results for problem **P** are presented in Table 3; **HKE** is **HGPC** algorithm with **KE** solver in Step 1, whereas **HDWE** has as iterative solver in Step 1 the **DWE** algorithm from Popa (2010). We notice that **DWE**, being sparsity pattern oriented, is more efficient (see Popa, 2010, and numerical experiments therein); moreover, it is a fully parallelizable method.

Tables 4 and 5 indicate the solutions obtained for the inconsistent transportation problem **P**. We observe that **HKE** or **HDWE** algorithm solution is more reliable (for a practical view point).

4. Conclusions

In the present paper we considered some iterative solvers for inconsistent systems of linear inequalities. We started with the original Han algorithm from Han (1980), and the iterative projection-based approximation with Kaczmarz Extended (KE) algorithm from Carp et. (2015). Then we proposed and theoretically analyzed the replacement of the KE method with a general projection-based algorithm. This allowed us finally to also consider, beside KE, the more sparsity oriented DWE algorithm from Popa, 2010. We then performed numerical experiments and comparisons with both projection-based solvers, KE and DWE, on an unbalanced and inconsistent transport model problem.

Table 3: Results for problem **P**

Algorithm	**cost**	$\|\|(Ax-b)_+\|\|$
HKE	15336	38.7529
HDWE	15337	38.7002
Simplex	31235*	145.0241

where * denotes that the Simplex algorithm failed to solve the problem, returning instead a result that minimizes *the worst case constraint violation* (see Vanderbei, 2001).

Table 4. The values $x_{ij}, i=1,\ldots,7, j=1,\ldots,7$ for the solution of problem **P** with **HKE** or **HDWE** algorithm

i j	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	0	144	530	0	0	387	0
2	0	0	0	360	0	0	0
3	0	0	0	89	232	0	159
4	445	166	0	0	0	0	0
5	0	0	0	0	0	0	610
6	0	0	0	0	490	0	0
7	0	0	0	0	28	433	0

Table 5. The values $x_{ij}, i=1,\ldots,7, j=1,\ldots,7$ for the solution of problem **P** with Simplex algorithm

i j	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	0	0	0	0	0	415	635
2	0	0	0	0	0	350	0
3	0	0	0	0	405	65	0
4	0	0	0	245	355	0	0
5	0	0	385	215	0	0	0
6	0	320	155	0	0	0	0
7	450	0	0	0	0	0	0

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