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OPTIMISATION OF POLYNOMIAL RAILWAY TRANSITION CURVES OF EVEN DEGREES

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Abstract: *This paper represents new results obtained by its authors while searching for the proper shape of polynomial railway transition curves (TCs). The search for the proper shape means the evaluation of the curve properties based on chosen dynamical quantities and generation of such shape with use of mathematically understood optimisation methods. The studies presented now and in the past always had got a character of the numerical tests. For needs of this work advanced vehicle model, dynamical track-vehicle and vehicle-passenger interactions, and optimisation methods were exploited. In this software complete rail vehicle model of 2-axle freight car, the track discrete model, and non-linear description od wheel-rail contact are used. That part of the software, being vehicle simulation software, is combined with library optimisation procedures into the final computer programme. The main difference between this and previous papers by the authors are the degrees of examinated polynomials. Previously they tested polynomial curves of odd degrees, now they focus on TCs of 6th, 8th and 10th degrees with and without curvature and superelevation ramp tangence in the TC's terminal points. Possibility to take account of fundamental demands (corresponding values of curvature in terminal points) concerning TC should be preserved. Results of optimisation are compared both among themselves and with 3rd degree parabola.*
The aim of present article is to find the polynomial TCs' optimum shapes which are determined by the possible polynomial configurations. Only one dynamical quantities being the results of simulation of railway vehicle advanced model is exploited in the determination of quality function (QF1). This is: minimum of integral of vehicle body lateral acceleration.

Key words: *polynomial railway transition curves, computer simulation, optimization*

1. Introduction

In recent works (Woźnica, 2012; Zboiński, 2012), authors of this article showed, that for the polynomial transition curves of odd degrees (5th, 7th, 9th and 11th) the best dynamical properties (the smallest values of QF1) have curves with the biggest possible number of their terms. For curves of 5th degree the number of terms was 3, for curves of 7th degree – 5, for curves of 9th degree – 7, and corresponding for curves of 11th degree – 9. In this context, serious difference between curves of lower and higher degrees was revealed. The curves of 5th and 7th degree had worse dynamical properties than 3rd degree parabola, whereas the curves of 9th and 11th degree possess such properties better than 3rd degree parabola. It was also shown that use of polynomial TCs in railway conditions could be an advantage. This can only be achieved, however as mentioned, for the curves of high degree and preferably with the maximum number of the terms. The best dynamical properties of such TCs were also confirmed through the simulation results representing vehicle body lateral displacements and accelerations. Such numbers of the terms correspond to the quite fundamental geometrical demand – curvature and superelevation equal to 0 at the initial points and $1/R$ and, respectively, H at the end points. This conclusion was true for all polynomial degrees, from 5th to 11th ones. It was also manifested univocally that the greater degree of the polynomial and number of its terms, the greater flexibility of TCs in terms of their shape. It was shown explicitly that use of polynomial TCs can be an advantage in the railway conditions. So motivation for the current studies arose from earlier results by the authors and the wish the research of polynomial of even degrees (6th, 8th and 10th) with maximum numbers of terms.
An increase in number of the publications that deal with transition curves, both the railway and the road ones can indeed be observed (Ahmad and Ali, 2008;

Ahmad et al. , 2007; Droździel and Sowiński, 2006; Fischer, 2009; Habib and Sakai, 2003; Koc and Mieloszyk, 1987; Koc and Radomski, 1985; Kuvfer, 2000; Li et al. , 2006; Long et al. , 2010, Pombo and Ambrosio, 2003; Tanaka, 1935; Tari and Baykal, 2005; Woźnica, 2012; Zboiński, 2012; Zboiński, 2004). The same touches railway dynamics e.g. Dusza (2014) or Kardas-Cinal (2014). Also some qualitative change in content of works concerning TCs can be noticed. It consists in attempts to diverge from the standard and to look for new, more modern methods of evaluating properties of TCs. Despite these, some earlier visible limitations of those works still exist, in present authors opinion. Namely, the analysis is rather rare which takes account of advanced dynamics of whole vehicle-track system. Present authors do not know method applied in practice (approved as a design tool), which uses complete dynamical model of vehicle in formation of railway TCs. Many methods in use represent traditional approach. They are based on the traditional criteria and often on very simple vehicle model. The authors failed to find publications that exploit directly mathematically understood optimisation methods in formation of TCs, basing on objective functions calculated as a result of numerical simulations. There are some works where selected quantities of interest, rather than the shape of the TC itself, are optimised instead, e.g. Kuvfer (2000).

In many works, also the recent ones, approach to the track-vehicle interactions is traditional, e.g. Esveld (1989). It is limited to discussing the vehicles jointly and studying the selected effects (quantities) in the car body. In such works the traditional criteria of 3-dimensional TCs' formation are in use. They demand from the physical quantities that characterise effects on a passenger and eventually on a cargo not to exceed values that are acknowledged as acceptable (Esveld, 1989). The corresponding relations refer to: unbalanced lateral acceleration $a \leq a_{lim}$, velocity of the a change $\psi \leq \psi_{lim}$, and velocity of wheel vertical rise along the superelevation ramp $f \leq f_{lim}$. Some up-to-date works extend these criteria with additional quantities and search for their courses. Such a quantity is the second time derivative of a. In case of the courses (of the a first and second derivatives most often), the continuity (no abrupt change in values), differentiability (no bends) and so on are demanded. Despite that extension, such criteria do not take account of the dynamical properties of particular vehicle, including track-vehicle interactions in particular conditions, or effects on vehicle bogie. These properties are quite different than those assumed in the traditional criteria. In these criteria the track has infinite stiffness and no geometrical irregularities, whereas vehicle is represented by a single rigid body or a particle.

2. Method of the analysis used for needs of current research

2.1. The object, its model, and the corresponding model

In order to demonstrate the method used in the research three elements will be discussed. The first element is railway vehicle, its model, and the simulation software. The second is the software in general. The last one touches the optimisation method, quality function (QF1) implemented in the program, and applied initial shapes of the TCs.

In order to make analysis easier relatively simple object and its model were utilised. This model represents 2-axle HSFV1 freight car of the average values of parameters. It is the same model of the system as used in the earlier studies by present authors (Woźnica, 2012; Zboiński, 2012; Zboiński 2004). Its structure is shown in Fig. 1c. It is supplemented with discrete models of vertically and laterally flexible track shown in Fig 1a and 1b, respectively. Linearity of the vehicle suspension was assumed. So, linear stiffness and damping elements in vehicle suspension were applied. The same concerns the track models. Here also linear stiffness and damping elements were applied. One can find all parameters of the used models in Zboiński (2012).

Vehicle model is equipped with a pair of wheel/rail profiles that corresponds to the real ones. That is a pair of the nominal (i.e. unworn) S1002/UIC60 profiles that are used all over the Europe. Non-linear geometry of this pair is introduced into the model in a form of table with the contact parameters. In order to calculate non-linear tangential contact forces between wheel and rail well known FASTSIM program by J.J. Kalker was applied. Normal forces in the contact are not constant but influenced by both the geometry and the dynamical effects that make value of a wheelset vertical load variable.

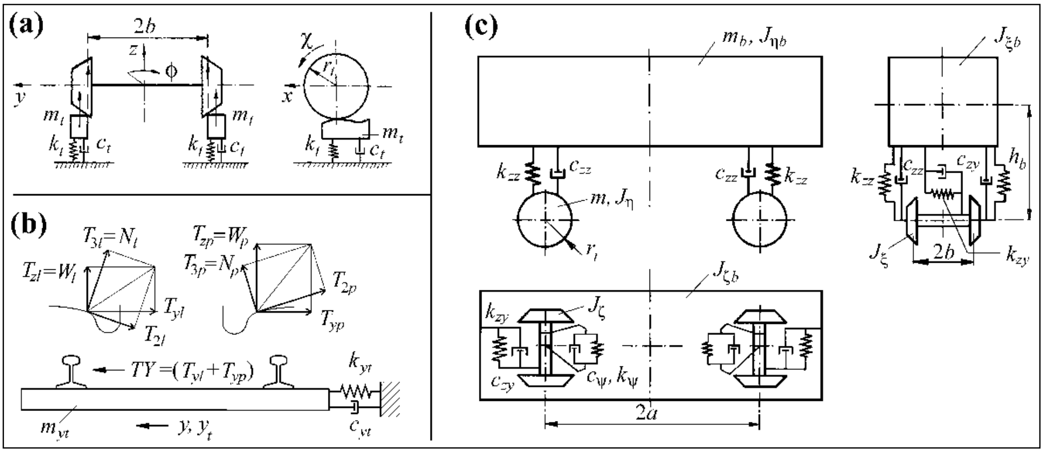

Fig. 1. System's nominal model: (a) track vertically, (b) track laterally, (c) vehicle

Generalised approach to the modelling was used, as explained in e.g. Zboiński (2012). Basically, dynamics of relative motion is used in that approach. This means that description of motion (dynamics) is relative to track-based moving reference frames. Dynamical equations of motion are equations of relative motion with terms depending on motion of the reference frames explicitly recorded. None of such terms is omitted in the equations. According to this method, the kinematic type non-linearities arising from rotational motions of bodies within our MBS model are taken into account, too. The term generalised refers first of all to the generalised conditions of motion. So, the same generalised vehicle model describes vehicle dynamics in any conditions, i.e. in straight track (ST), circular curve (CC), and TC sections. The routes composed of such sections can also be analysed.

The route (section) of interest is characterised in the method by shape of the track centre line which is the general space (3-dimensional) curve. In railway systems such 3-dimensional objects are TCs with their superelevation ramps. A necessary condition to apply the method is description of the curves (sections) by parametric equations, with the curve's current length l as the parameter. The cases of CC and ST are treated in the method as the special cases of 2-dimensional and 1-dimensional geometrical objects, respectively. Such an approach was described in Zboiński (2012).

An important element in the method is description of kinematics of the track-based moving reference frames. Their motion comes out directly from the track centre line shape. The applied method of determination of the kinematical quantities on the basis of the parametric equations is presented in Zboiński (2012).

2.2. The optimization method and objective functions

The optimization problem which is solved in the current studies is to find the A_i polynomial coefficients that define TC's shape. Type of a TC chosen for optimisation is the polynomial TC of any degree n, $n\geq 4$. It is defined by Eqs. (1)-(4) that are related to space curve parametric equations:

$$y=\frac{1}{R}\left(\begin{array}{l}\frac{A_n l^n}{l_0^{\,n-2}}+\frac{A_{n-1} l^{n-1}}{l_0^{\,n-3}}+\\ +\frac{A_{n-2} l^{n-2}}{l_0^{\,n-4}}+\frac{A_{n-3} l^{n-3}}{l_0^{\,n-5}}+\\ +.....+\frac{A_4 l^4}{l_0^{\,2}}+\frac{A_3 l^3}{l_0^{\,1}}\end{array}\right), \tag{1}$$

$$k=\frac{d^2y}{dl^2}=\frac{1}{R}\left[\begin{array}{l}n(n-1)\frac{A_n l^{n-2}}{l_0^{\,n-2}}+\\ +(n-1)(n-2)\frac{A_{n-1} l^{n-3}}{l_0^{\,n-3}}+\\ +.....+3\cdot 2\frac{A_3 l^1}{l_0^{\,1}}\end{array}\right], \tag{2}$$

$$h = H\left[\begin{array}{l} n\left(n-1\right)\dfrac{A_n l^{n-2}}{l_0^{\,n-2}} + \\ +\left(n-1\right)\left(n-2\right)\dfrac{A_{n-1} l^{n-3}}{l_0^{\,n-3}} + \\ +.....+4\cdot 3\dfrac{A_4 l^2}{l_0^{\,2}}+3\cdot 2\dfrac{A_3 l^1}{l_0^{\,1}} \end{array}\right], \tag{3}$$

$$i = \frac{dh}{dl} = H\left[\begin{array}{l} n\left(n-1\right)\left(n-2\right)\dfrac{A_n l^{n-3}}{l_0^{\,n-2}} + \\ +\left(n-1\right)\left(n-2\right)\left(n-3\right)\dfrac{A_{n-1} l^{n-4}}{l_0^{\,n-3}} + \\ +.....+3\cdot 2\cdot 1\dfrac{A_3 l^0}{l_0^{\,1}} \end{array}\right], \tag{4}$$

where *y*, *k*, *h*, and *i* define curve lateral co-ordinate, curvature, superelevation, and inclination of superelevation ramp, respectively. The *R*, *H*, l_0, and *l* define curve minimum radius (at its end), maximum susperelevation (at the curve end), total curve length, and curve current length, respectively. The A_i are polynomial coefficients ($i = n, n$-1,…., 4, 3) while *n* is polynomial degree. Here, *n*=6, 8 and 10. Number of the polynomial terms (terms in Eqs. (1)-(4)) must not be smaller than 2. On the other hand the smallest degree n_{min} of the last term in Eq. (1) must be $n_{min} \geq 3$. Such definition of the curves gives possibility of proper *k* and *h* values at TCs terminal points. They should equal to 0 at the initial points and to 1/*R* and *H* at the end points. Note, that values for both always equal to 0 for *l*=0. In order to ensure 1/*R* and *H* values for *l*=*L*, normalisation of the coefficients is necessary, such as in Woźnica (2012). Finally, coefficients A_i' are obtained which satisfy constraints imposed on their values. The problem just formulated is a classical formulation of the static constrained optimisation. It is realised with the library procedure that utilises moving penalty function algorithm combined with Powell's method of conjugate directions.

For needs of current paper authors utilised one quality function (QF1) marked number 1. This function concerns a minimisation of integral of vehicle body lateral acceleration:

$$QF_1 = L_C^{-1}\int_0^{L_C} |\,\ddot{y}_b\,|\,dl\,, \tag{5}$$

where $\ddot{y}_b$ lateral acceleration of body vehicle, respectively, and L_C – whole TC and the adjacent CC of 100 m length.

The difficulty of the problem solution consists in quite complex form and way to determine the objective function (quality function). This function is calculated as a result of the numerical simulation of motion of the dynamical mechanical system as described in Subsection 2.1. The main steps during calculation of the objective function are: generation of the new shape of TC, calculation of the kinematical quantities (velocities and accelerations) that depend on this new shape, and solution of the corresponding 2nd degree ordinary differential equations (ODEs) set. Note that here, this system of equations describes dynamical system of 18 degrees of freedom.

2.3. General look at the Software

Scheme of the software used in optimisation TCs shape is shown in Figure 2. The major objects within this scheme are two iteration loops visible there. The first is the integration loop. This loop is stopped when distance l_{lim}, being the length of route (usually compound route ST, TC and CC or CC, TC and ST), is reached the model. The second is the optimisation process loop. It is stopped when number of iterations reaches limit value i_{lim}. This value means that i_{lim} simulations of vehicle motion have to be performed in order to stop optimisation process. In the calculations done so far i_{lim}=700 was used as standard value. If the optimum solution is reached earlier, i.e. for $i<i_{lim}$, then the optimisation process stops automatically and the corresponding results are recorded. When no optimum solution is reached for i_{lim}=700, then this value has to be increased manually, while the process has to be repeated.

Usually calculation time of the single process on the PC computer with Inter Core 2 Duo 2GB processor lasted from 30 to 80 minutes. No calculation times longer than 80 minutes happened, so far.

2.4. Kinematical properties of the polynomial transition curve of any order

In order to reflect precisely kinematical properties of the TC chosen in the previous subsection the components of angular velocity ω and acceleration ε of transportation must be known. These are quantities that represent TC shape in the dynamical model.

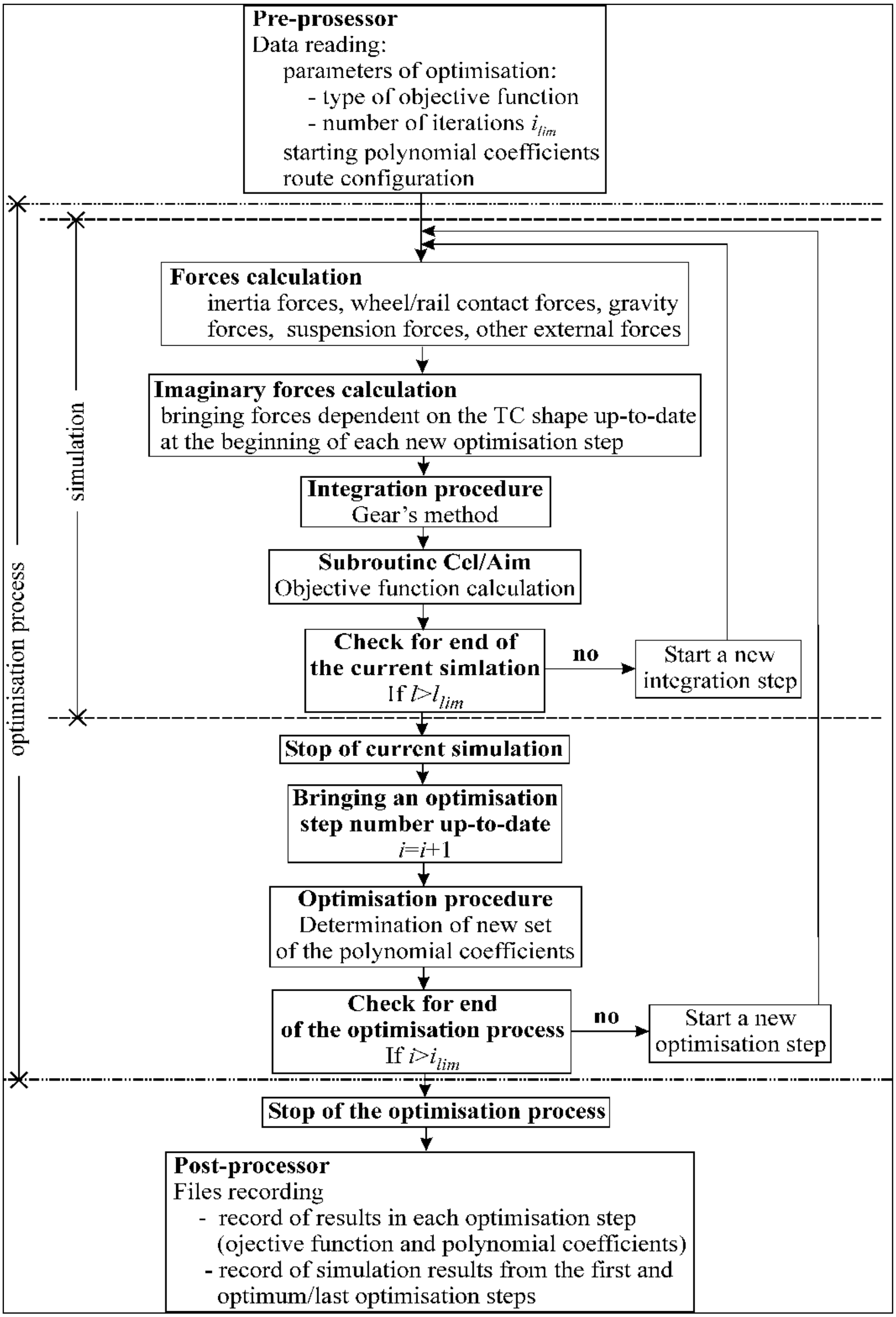

Fig. 2. General scheme of the software to optimise transition curves' shape

As mentioned earlier, the general method of determination of these components is presented in Zboiński (2012). Fundamental relationships, invoked from Zboiński (2012), that define these components in the natural (moving trihedral) system are as follows:

$$\boldsymbol{\omega} = \mathbf{t}\cdot\omega_t + \mathbf{n}\cdot\omega_n + \mathbf{b}\cdot\omega_b \cong \mathbf{t}\cdot\left(\mathrm{d}\gamma/\mathrm{d}t\right) + \mathbf{b}\cdot vk \tag{9}$$

$$\begin{aligned}\boldsymbol{\varepsilon} &= \mathbf{t}\cdot\varepsilon_t + \mathbf{n}\cdot\varepsilon_n + \mathbf{b}\cdot\varepsilon_b = \\ &= \mathbf{t}\cdot\left(\frac{\mathrm{d}\omega_t}{\mathrm{d}t}\right) + \mathbf{n}\cdot\left(vk\omega_t + v\tau\omega_b\right) + \mathbf{b}\cdot\left(\frac{\mathrm{d}\omega_b}{\mathrm{d}t}\right) = \\ &= \mathbf{t}\cdot\left(\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}\right) + \mathbf{n}\cdot\left[vk\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) + v^2\tau k\right] + \mathbf{b}\cdot\left[\frac{\mathrm{d}(vk)}{\mathrm{d}t}\right]\end{aligned} \tag{10}$$

where ***t***, ***n***, and ***b*** are versors of the natural system axes, γ is angle corresponding to superelevation *h*, and *v* is vehicle (variable) speed. It is seen in Eqs. (9) and (10) that one must know γ, *v*, *k*, and τ as well as some derivatives of these quantities in order to calculate components ω_t, ω_n, ω_b and ε_t, ε_n, ε_b. It can also be expected, when looking at Eqs. (9) and (10), that full analytical form of the components will be the complex one. That is why we do not tend to present full analytical form of the ω_t, ω_n, ω_b and ε_t, ε_n, ε_b in this paper. Instead, we will present form of the factors and terms that are directly used by us while calculating values of the components in the numerical model (code of the software).

Let us start with the angle γ. According to Zboiński (2012) the following formula holds:

$$\gamma = \arcsin\left[z(l)/b\right] \cong z(l)/b \tag{11}$$

where *b* is a half of the track gauge. The approximate version of (11) holds for small values of the angle. Note, that in real track γ ≤ 6°. Consequently:

$$\gamma \cong \frac{H}{2b}\left[\begin{array}{l} n(n-1)\dfrac{A'_n l^{n-2}}{l_0^{\,n-2}} + \\ +(n-1)(n-2)\dfrac{A'_{n-1} l^{n-3}}{l_0^{\,n-3}} + \\ +.....+5\cdot 4\dfrac{A'_5 l^3}{l_0^{\,3}} + 4\cdot 3\dfrac{A'_4 l^2}{l_0^{\,2}} \end{array}\right] \tag{12}$$

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dl}\cdot\frac{dl}{dt} = v\cdot\frac{d\gamma}{dl} = \frac{v}{2b}\cdot\frac{dh}{dl} = \frac{v}{2b}\cdot i =$$

$$= \frac{vH}{2b}\left[\begin{array}{l} n(n-1)(n-2)\dfrac{A_n l^{n-3}}{l_0^{\,n-2}} + \\ +(n-1)(n-2)(n-3)\dfrac{A_{n-1} l^{n-4}}{l_0^{\,n-3}} + \\ +.....+5\cdot 4\cdot 3\dfrac{A_5 l^2}{l_0^{\,3}} + 4\cdot 3\cdot 2\dfrac{A_4 l^1}{l_0^{\,2}} + \\ +3\cdot 2\cdot 1\dfrac{A_3 l^0}{l_0^{\,1}} \end{array}\right] \tag{13}$$

$$\begin{aligned}\frac{d^2\gamma}{dt^2} &= d\left(\frac{vi}{2b}\right)/\,dt = \frac{1}{2b}\cdot\frac{d(vi)}{dt} = \\ &= \frac{1}{2b}\left(\frac{dv}{dt}\cdot i + v\cdot\frac{di}{dt}\right) == \frac{1}{2b}\left(a\cdot i + v\cdot\frac{dl}{dt}\cdot\frac{di}{dl}\right) = \\ &= \frac{1}{2b}\left(a\cdot i + v^2\cdot\frac{di}{dl}\right)\end{aligned} \tag{14}$$

where in Eq. (14) the acceleration $a = \mathrm{d}v/\mathrm{d}t$ can be assumed as known. It is like that because change of the *v* and *l* in time must be known in case we want to consider the relative kinematics. Let us recall that *v* is velocity of the transportation system origin. The *i* is defined by (4) and d*i*/d*l* is given below.

$$\frac{di}{dl} = H\left[\begin{array}{l} n(n-1)(n-2)(n-3)\dfrac{A'_n l^{n-4}}{l_0^{\,n-2}} + \\ +(n-1)(n-2)(n-3)(n-4)\dfrac{A'_{n-1} l^{n-5}}{l_0^{\,n-3}} \\ +.....+5\cdot 4\cdot 3\cdot 2\dfrac{A'_5 l^1}{l_0^{\,3}} + 4\cdot 3\cdot 2\cdot 1\dfrac{A'_4 l^0}{l_0^{\,2}} \end{array}\right] \tag{15}$$

Now, let us discuss curvature *k* that is also present in Eqs. (9) and (10). We have in fact two options of its calculation. The first is direct use of (2). It is a simplified formula for the *k*. It generates small errors in general. Discussion of such errors' value is done in Zboiński (2012). The other option is use of the non-simplified formula that holds for any 3-dimensional curve represented by the parametric equations. It is as follows:

$$k = \sqrt{\left(\mathrm{d}^2 x/\mathrm{d}l^2\right)^2 + \left(\mathrm{d}^2 y/\mathrm{d}l^2\right)^2 + \left(\mathrm{d}^2 z/\mathrm{d}l^2\right)^2} \qquad (16)$$

It is seen from Eq. (7) that first of the terms under the square root sign equals 0. Using Eqs. (3), (4) and (7) the missed term for the *z* co-ordinate can be calculated as:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}l^2} = \frac{1}{2}\cdot\frac{\mathrm{d}^2 h}{\mathrm{d}l^2} = \frac{1}{2}\cdot\frac{\mathrm{d}i}{\mathrm{d}l} \qquad (17)$$

where d*i*/d*l* is given in Eq. (15). In our further calculations and in the software used to generate the simulation and optimisation results the first option is used.

The next to discuss is torsion τ of the curve that is also necessary to determine the kinematical components from Eqs. (9) and (10). Let us start with the general formula for the torsion known in the differential geometry.

$$\tau = \frac{1}{k^2}\cdot\begin{vmatrix} (\mathrm{d}x/\mathrm{d}l) & (\mathrm{d}y/\mathrm{d}l) & (\mathrm{d}z/\mathrm{d}l) \\ (\mathrm{d}^2x/\mathrm{d}l^2) & (\mathrm{d}^2y/\mathrm{d}l^2) & (\mathrm{d}^2z/\mathrm{d}l^2) \\ (\mathrm{d}^3x/\mathrm{d}l^3) & (\mathrm{d}^3y/\mathrm{d}l^3) & (\mathrm{d}^3z/\mathrm{d}l^3) \end{vmatrix} \qquad (18)$$

In order to avoid unnecessary calculation one can note that first column of the determinant in (18) equals 1, 0, and 0. Then (18) can be recorded as follows:

$$\begin{aligned} \tau &= \frac{1}{k^2}\cdot\begin{vmatrix} 1 & (\mathrm{d}y/\mathrm{d}l) & (\mathrm{d}z/\mathrm{d}l) \\ 0 & (\mathrm{d}^2y/\mathrm{d}l^2) & (\mathrm{d}^2z/\mathrm{d}l^2) \\ 0 & (\mathrm{d}^3y/\mathrm{d}l^3) & (\mathrm{d}^3z/\mathrm{d}l^3) \end{vmatrix} = \\ &= \frac{1}{k^2}\left[\left(\frac{\mathrm{d}^2y}{\mathrm{d}l^2}\right)\cdot\left(\frac{\mathrm{d}^3z}{\mathrm{d}l^3}\right) - \left(\frac{\mathrm{d}^2z}{\mathrm{d}l^2}\right)\cdot\left(\frac{\mathrm{d}^3y}{\mathrm{d}l^3}\right)\right] = \qquad (19) \\ &= \frac{1}{k^2}\left[k\cdot\left(\frac{\mathrm{d}^3z}{\mathrm{d}l^3}\right) - \frac{1}{2}\frac{\mathrm{d}i}{\mathrm{d}l}\cdot\left(\frac{\mathrm{d}^3y}{\mathrm{d}l^3}\right)\right] \end{aligned}$$

In order to make use of last line in Eq. (19), the two expressions in round brackets have to be calculated. Looking at Eqs. (15) and (17) one can note that

$$\begin{aligned} \frac{\mathrm{d}^3z}{\mathrm{d}l^3} &= \frac{\mathrm{d}\left(\frac{1}{2}\cdot\frac{\mathrm{d}i}{\mathrm{d}l}\right)}{\mathrm{d}l} = \\ &= \frac{H}{2}\begin{bmatrix} n(n-1)(n-2)(n-3)(n-4)\dfrac{A'_n l^{n-5}}{l_0^{\,n-2}} + \\ +(n-1)(n-2)(n-3)(n-4)(n-5)\dfrac{A'_{n-1} l^{n-6}}{l_0^{\,n-3}} + \\ +.....+5\cdot4\cdot3\cdot2\cdot1\dfrac{A'_5 l^0}{l_0^{\,3}} \end{bmatrix} \end{aligned} \qquad (20)$$

Taking account of Eqs. (2)-(4) one can write down that:

$$\frac{\mathrm{d}^3y}{\mathrm{d}l^3} = \frac{1}{RH}\cdot i \qquad (21)$$

where *i* is determined with Eq. (4).

The last term that need to be calculated while determining the components of angular velocity and acceleration of transformation is the last term in Eq. (10). It defines the ε_b component. One can perform the following manipulation for it:

$$\begin{aligned} \frac{\mathrm{d}(vk)}{\mathrm{d}t} &= \left(\frac{\mathrm{d}v}{\mathrm{d}t}\cdot k + v\cdot\frac{\mathrm{d}k}{\mathrm{d}t}\right) = \left(a\cdot k + v\cdot\frac{\mathrm{d}l}{\mathrm{d}t}\cdot\frac{\mathrm{d}k}{\mathrm{d}l}\right) = \\ &= \left(a\cdot k + v^2\cdot\frac{\mathrm{d}k}{\mathrm{d}l}\right) = \left(a\cdot k + v^2\cdot\frac{\mathrm{d}^3y}{\mathrm{d}l^3}\right) \end{aligned} \qquad (22)$$

where $\mathrm{d}^3y/\mathrm{d}l^3$ is given through Eq. (21).

This way the components ω_t, ω_n, ω_b and ε_t, ε_n, ε_b became determinate. The presented formulae are used in the numerical code we elaborated. Note, that so defined components concern the natural system. If one needs the components in the transportation system then they must be transformed. It is done with use of the direction cosine matrix between these co-ordinate systems. The matrix and transformation itself are presented in detail in Zboiński (2012).

3. Results of the studies

3.1. On polynomial TCs of even degrees some informations

Each polynomial of even degree has two different standard transition curves (INI) and this property differs such polynomials from polynomials of odd degrees (where is only one standard TC). In this case, the functions of inclination of superelevation ramp – formula (3) – are symmetrical about a vertical axis passing through the point $l_0/2$. The method of receiving two standard TC for even degrees is presented in Woznica (2012). The full list of standard TC of 6th, 8th and 10th is demonstrated in Tab. 1.

For each polynomial curve (also odd degree) geometrical demands were imposed that one wants or does not want to take into account. Possible combinations of coefficients are shown in Tab. 2. In this paper, as mentioned, authors focused only on curves with maximum number of terms.

Each TC has minimal length which is calculated in accordance with the method presented in Koc and Radomski (1985). This minimal length arises from the fact that two values are not allowed to be exceeded.

Table 1. Standard (initial) TCs of 6th, 8th and 10th degrees

Degree of polynomial	**Two standard (initial) TCs (INI)**
6th	$y_1=\frac{1}{R}\left(-\frac{l^6}{10l_0^4}+\frac{1}{5}\frac{l^5}{l_0^3}\right)$ $y_2=\frac{1}{R}\left(\frac{l^6}{10l_0^4}-\frac{2}{5}\frac{l^5}{l_0^3}+\frac{1}{2}\frac{l^4}{l_0^2}\right)$
8th	$y_1=\frac{1}{R}\left(+\frac{5}{28}\frac{l^8}{l_0^6}-\frac{4}{7}\frac{l^7}{l_0^5}+\frac{1}{2}\frac{l^6}{l_0^4}\right)$ $y_2=\frac{1}{R}\left(-\frac{5}{28}\frac{l^8}{l_0^6}+\frac{6}{7}\frac{l^7}{l_0^5}-\frac{3}{2}\frac{l^6}{l_0^4}+\frac{l^5}{l_0^3}\right)$
10th	$y_1=\frac{1}{R}\left(-\frac{7}{18}\frac{l^{10}}{l_0^8}+\frac{5}{3}\frac{l^9}{l_0^7}+ -\frac{5}{2}\frac{l^8}{l_0^6}+\frac{4}{3}\frac{l^7}{l_0^5}\right)$ $y_2=\frac{1}{R}\left(\frac{7}{18}\frac{l^{10}}{l_0^8}-\frac{20}{9}\frac{l^9}{l_0^7}+ +5\frac{l^8}{l_0^6}-\frac{16}{3}\frac{l^7}{l_0^5}+\frac{7}{3}\frac{l^6}{l_0^4}\right)$

Table 2. Possible polynomial configurations for different geometrical demands

Type of demand ⇒ ___________ Polynomial degree (terms number in the initial polynomial) ⇓	**Demand IDZ=1** (proper values of *r* and *h* in TCs' terminal points) **Number of terms**	**Demand IDZ=2** (tangence of *r* and *h* functions at TCs' terminal points) **Number of terms**	**Demand IDZ=3** (tangence of *h* slope, i.e. of *i*, at TCs' terminal points) **Number of terms**
6th (IWI=2, 3)	IW=2; IW=3; IW=4	IW=2(single curve); IW=3	-
8th (IWI=3, 4)	IW=2; IW=3; IW=4; IW=5; IW=6	IW=2(single curve); IW=3; IW=4; IW=5;	IW=3(single curve); IW=4
10th (IWI=4, 5)	IW=2; IW=3; IW=4; IW=5; IW=6; IW=7; IW=8	IW=2(single curve); IW=3; IW=4; IW=5; IW=6; IW=7	IW=3(single curve); IW=4; IW=5; IW=6

First one is the velocity of the unbalanced lateral acceleration change ψ and second one is the velocity of wheel vertical rise along the superelevation ramp *f*. Minimum lengths for two longitudinal velocities v=24.26 m/s and v=30.79 m/s (corresponding to lateral acceleration a equal to 0 and 0.6 m/s^2, radius of circular arc R=600 m and superelevation H=0.15 m), velocity of wheel vertical rise along the superelevation ramp f=56 mm/s, and velocity of the unbalanced lateral acceleration change ψ=1 m/s^3 are presented in Tab. 3. For needs of this work authors always took a greater length calculated for both conditions ($l_{f\,min}$).

Table 3. Minimal lenghts of TCs

Degree of polyn.	$l_{f\,min}$ **[m]** - v=24.26 m/s **(**v=30.79 m/s**)**	$l_{\psi\,min}$ **[m]** - v=24.26 m/s **(**v=30.79 m/s**)**
6th	115.47 (146.59)	0 (32.83)
8th	134.75 (171.06)	0 (38.31)
10th	152.71 (193.87)	0 (43.42)

3.2. Result of optimization and dynamical simulations

Graphical representation of the results are polynomial TCs of 6th, 8th and 10th degrees. Three geometrical demand are taken into account in the presented article, namely - IDZ=1, IDZ=2 and IDZ=3 (see Tab. 2). Polynomial configurations are not limited to those with the maximum number of terms, as in Woźnica (2012) (see Table 2). As concerns configuration of test routes they are always composed of ST, TC, and CC. Lengths of ST sections are the same and equal 50 m. Similarly for CC, their lengths are the same and equal 100 m. Besides, single curve radius R and superelevation H for CC were considered. Their values were R=600 m and H=0.15 m. Two sorts of the TCs' parameters for such CC were considered and are presented below for each of the TC's degrees. For both sorts maximum velocity of wheel rise along superelevation ramp f=56 mm/s. Different for this sorts are velocities v that represent maximum an admissible vehicle velocity in curved track. It is determined for particular R, H and admissible unbalanced acceleration a_{lim} on the track level. Its value a_{lim}=0.0 and 0.6 m/s^2, respectively. They result in different TC's length as l_0 is a function of f, H, and v, of which v is different for both route sorts. In addition l_0 is a function of numerical coefficient proper for degree of the particular polynomial. Consequently six routes are considered and three different l_0 for them. On the other hand just two values of v were used. Differences between the routes exist for TCs only. Their parameters undefined so far are as follows: Route 1 (6th degree, v=30.79 m/s, l_0=146.59 m, INI=y_1, QF1, IDZ=1 (IW=4), IDZ=2 (IW=3)); Route 2 (6th degree, v=30.79 m/s, l_0=146.59 m, INI=y_2, QF1, IDZ=1 (IW=4), IDZ=2 (IW=3)); Route 3 (8th degree, v=24.26 m/s, L=134.75 m, INI=y_1, QF1, IDZ=1 (IW=6), IDZ=2 (IW=5), IDZ=3 (IW=4)); Route 4 (8th degree, v=24.26 m/s, L=134.75 m, INI=y_2, QF1, IDZ=1 (IW=6), IDZ=2 (IW=5), IDZ=3 (IW=4)); Route 5 (10th degree, v=24.26 m/s, l_0=152.71 m, INI=y_1, QF1, IDZ=1 (IW=8), IDZ=2 (IW=7), IDZ=3 (IW=6)); Route 6 (6th degree, v=30.79 m/s, l_0=146.59 m, INI=y_2, QF1, IDZ=1 (IW=8), IDZ=2 (IW=7), IDZ=3 (IW=6)).

Each of the routes is represented by its own group of four figures. Content of the figures in particular groups is analogous. So, first in the group is figure representing superelevation ramps h corresponding to all TCs' shapes tested in optimisation process. Note, that courses of the curvatures $1/r$ from the same process are identical with those for h. The only difference is scale of the vertical axis. The skew straight lines in that kind of figures are of no importance. They arise from recording results for all the shapes in a single file. Second in the group is figure representing curvature of the initial and optimised TCs. The third in the group is figure representing vehicle body lateral displacements for the initial, optimised and sometimes the parabolic TC. The forth is figure representing vehicle body lateral accelerations for the initial and optimised TC. Denotations INI and for example IDZ=1 (IW=6), IDZ=2 (IW=5), IDZ=3 (IW=4) mean results for the initial and optimised TCs with different geometrical demands. To make the figures better readable different line types were also applied. The solid line represents results after optimisation, while discontinuous line at the beginning of the optimisation process.

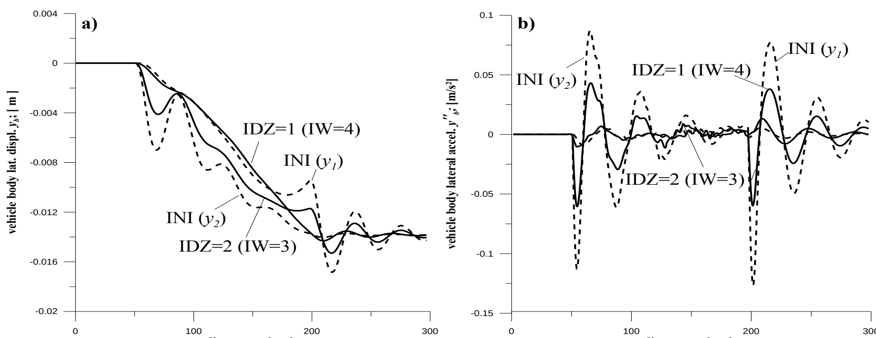

Fig. 4: Route 1, 2 - results of simulation for vehicle body for the initial and optimised TCs: a) lateral displacement, b) lateral acceleration

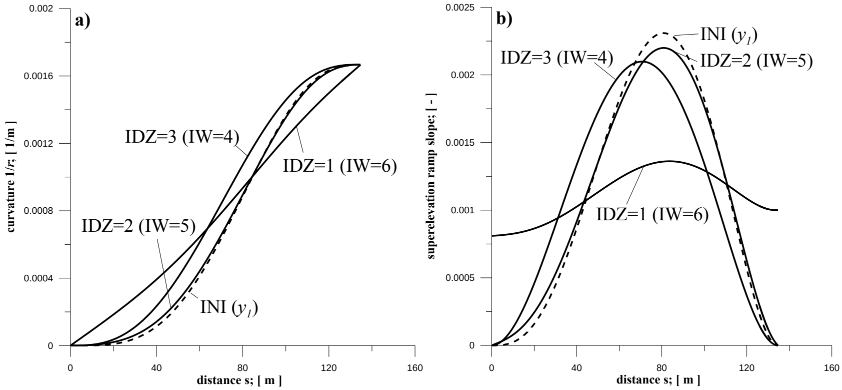

Fig. 5: <u>Route 3</u> - features of TCs: a) curvatures of the initial and optimised TCs, b) superelevation ramps' slopes

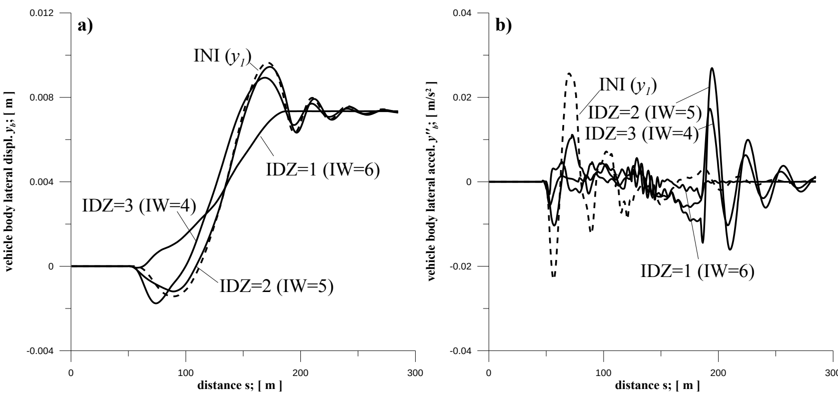

Fig. 6: <u>Route 3</u> - results of simulation for vehicle body for the initial and optimised TCs: a) lateral displacement, b) lateral acceleration

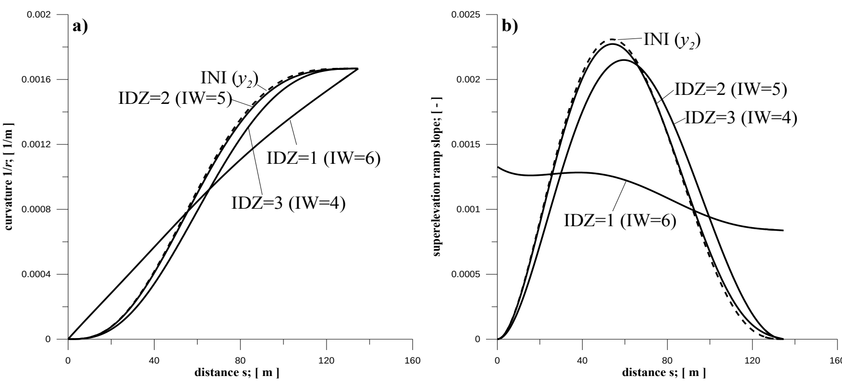

Fig. 7: <u>Route 4</u> - features of TCs: a) curvatures of the initial and optimised TCs, b) superelevation ramps' slopes

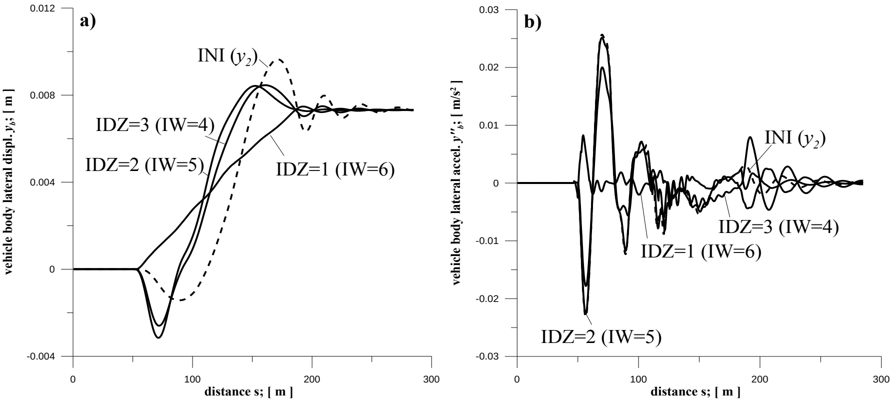

Fig. 8: Route 4 - results of simulation for vehicle body for the initial and optimised TCs: a) lateral displacement, b) lateral acceleration

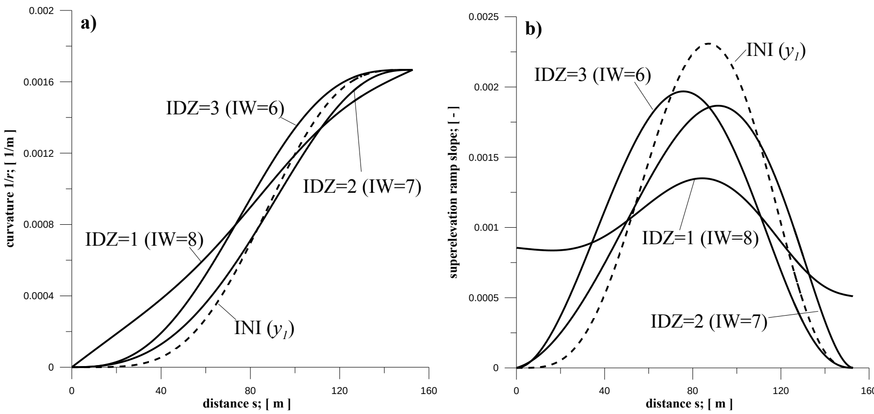

Fig. 9: Route 5 - features of TCs: a) curvatures of the initial and optimised TCs, b) superelevation ramps' slopes

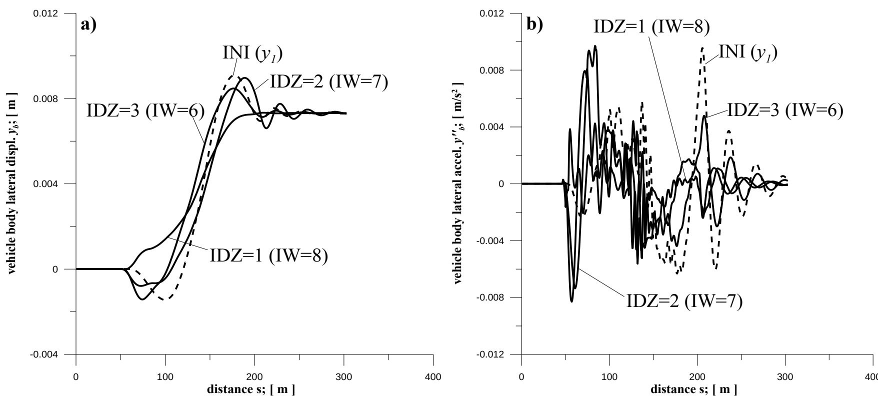

Fig. 10: Route 5 - results of simulation for vehicle body for the initial and optimised TCs: a) lateral displacement, b) lateral acceleration

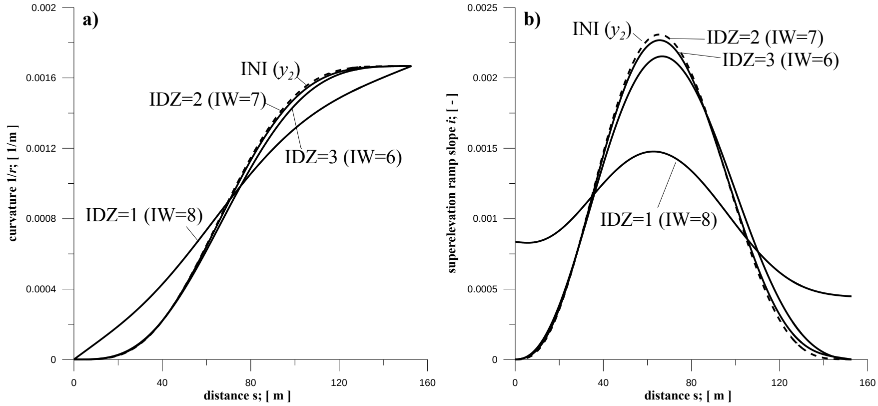

Fig. 11: Route 6 - features of TCs: a) curvatures of the initial and optimised TCs, b) superelevation ramps' slopes

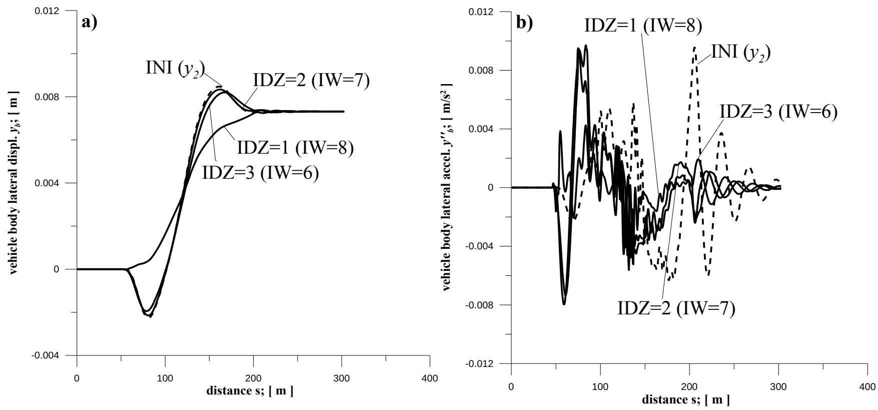

Fig. 12: Route 6 - results of simulation for vehicle body for the initial and optimised TCs: a) lateral displacement, b) lateral acceleration

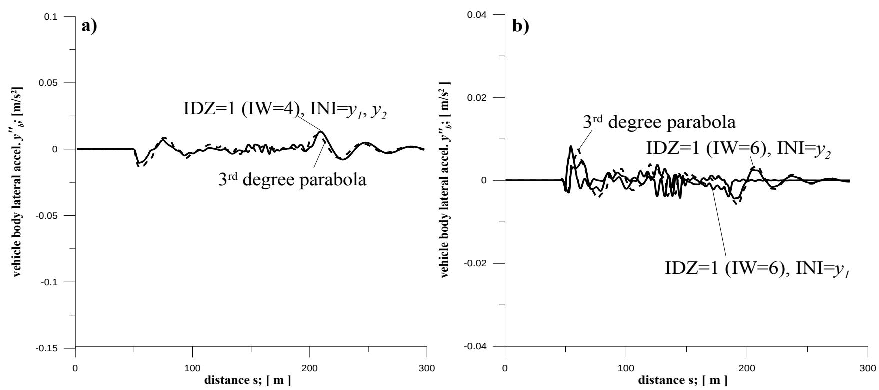

Fig. 13: Results of simulation for vehicle body (lateral acceleration) for the optimised TCs and 3rd degree parabola: a) 6th (Routes 1 and 2), b) 8th (Routes 3 and 4)

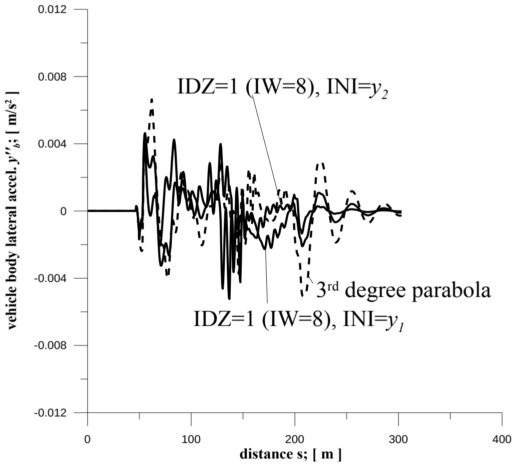

Fig. 14: Results of simulation for vehicle body (lateral acceleration) for the optimised TCs and 3rd degree parabola: 10th (Routes 5 and 6)

3.3. Discussion of the results obtained

Results of numerical calculations presented above can be divided into two categories. First is the category for IDZ=1 condition. Second is the category related to the IDZ=2 and IDZ=3 conditions. In case of the first category the optimum TCs' shapes are something between the initial curves and the linear shape for 3rd degree parabolic TC. In case of the second category optimum shapes have tangence of curvature in extreme points, look similar to the standard curvatures and superelevation ramps' slopes have „bell" shape.

Betterment in the system dynamical properties for the optimised TCs' shapes in comparison to the initial curves is confirmed by simulation results, the lateral displacements y_b and accelerations y''_b of vehicle body for all demands. It is the case for both these quantities in Figures 4, 6, 8, 10 and 12.

For Routes 1 and 2 different starting point lead to the very similar optimum solutions (Figs. 3 and 5). For this reason results for these Routes authors presented togehter. Optimum curvatures for the IDZ=1 demand hand't any tangence in extreme points. The direction of the curvatures' bend leads to convex (Figure 3a) shapes. Results for Routes from 3 to 6 are prestented in Figures from 5 to 12. In these case different starting point give different optimum solutions.

In table 4a and 4b authors by this work presented optimum coefficients A'_i and two percentage changes of value of quality function (which are ratio of value of quality function for optimum curves to value of quality function for initial curves and optimum curves for IDZ=1 demand to value of quality function for parabola 3rd degree). It was shown univocally that the polynomial TCs with the biggest possible number of their terms have the smallest values of their QFs, also in comparison with 3rd degree parabola. The corresponding numbers are 4, 6 and 8 for the 6th, 8th and 10th degrees, respectively (see Tab. 4a and b). Such numbers of the terms correspond to the quite fundamental geometrical demand IDZ=1 (see Tab. 2), while advanced demands IDZ=2 and 3 cannot be satisfied by these curves. This explicitly shows that use of the curves that satisfy the advanced demands is not the right way to improve dynamical properties of the vehicle-track system in TCs and adjacent part of CCs. This conclusion is true for all polynomial degrees, from 6th to 10th ones.

It was manifested univocally that the greater degree of the polynomial, this is not a greater level of flexibility of TCs in terms of their shape (as is the case for polynomial of odd degrees). It was shown explicitly that use of polynomial TCs can be an advantage in the railway conditions. This can only be achieved, however, for the curves of high degrees with the maximum number of their terms. Use of the advanced geometrical demands for polynomial TCs is a mistake, especially for the curves of the lower degrees.

Table 4. Results of optimization of shape of TCs

Optimized curve	Coefficients A_i' and percentage changes of value of quality function (value of quality function for optimum curve/value of quality function for initial curve and 3rd parab.)			
	IDZ=1	IDZ=2	IDZ=3	3rd degree parab.
6th (Route 1)	A_6' =-0.020839 A_5' =0.0521111 **19.96**% A_4' =0.0178607 A_3' =0.0614385	A_6' = 0.010731 A_5' =-0.132194 **90.15%** A_4' = 0.276828	-	**95.18%** (=3rd degree parab./ 6th IDZ=1)
6th (Route 2)	A_6' =-0.018988 A_5' =0.0463111 **19.91**% A_4' =0.0213801 A_3' =0.0644791	A_6' = 0.004420 A_5' =-0.113263 **89.97%** A_4' = 0.261052	-	**96.67%** (=3rd degree parab./ 6th IDZ=1)
8th (Route 3)	A_8' = 0.034045 A_7' =-0.108693 A_6' = 0.094386 **18.52**% A_5' = 0.001616 A_4' = 0.005625 A_3' = 0.121192	A_8' = 0.146147 A_7' =-0.470415 A_6' = 0.411117 **93.61%** A_5' =-0.006580 A_4' = 0.030942	A_8' = 0.03634 A_7' =-0.00251 A_6' =-0.29647 **77.33%** A_5' = 0.39824	**54.73%** (=3rd degree parab./ 8th IDZ=1)
8th (Route 4)	A_8' =-0.024362 A_7' = 0.115327 A_6' =-0.203640 **27.42%** A_5' = 0.154465 A_4' = 0.054394 A_3' = 1.477842	A_8' =-0.174106 A_7' = 0.833084 A_6' =-1.457150 **93.56%** A_5' =0.973003 A_4' =0.001234	A_8' =-0.08940 A_7' = 0.50046 A_6' =-1.00065 **89.26%** A_5' = 0.75032	**64.08%** (=3rd degree parab./ 8th IDZ=1)
10th (Route 5)	A_{10}' =-0.108791 A_9' = 0.467010 A_8' =-0.700877 A_7' = 0.373322 **34.06%** A_6' =-0.000717 A_5' = 0.000688 A_4' =-0.010553 A_3' = 0.145070	A_{10}' =-0.02006 A_9' =0.021598 A_8' =0.013993 A_7' =0.021770 **79.02%** A_6' =0.042751 A_5' =-0.16907 A_4' =-0.02901	A_{10}' =-0.10427 A_9' = 0.40720 A_8' =-0.55506 A_7' = 0.40916 **77.85%** A_6' =-0.48233 A_5' = 0.47174	**68.22%** (=3rd degree parab./ 10th IDZ=1)
10th (Route 6)	A_{10}' = 0.13171 A_9' =-0.75237 A_8' = 1.69408 A_7' =-1.80842 **38.86%** A_6' = 0.79043 A_5' = 0.00028 A_4' =-0.01310 A_3' =-0.14186	A_{10}' =0.37235 A_9' =-2.12806 A_8' = 4.78816 A_7' =-5.10737 **97.78%** A_6' = 2.234470 A_5' =-0.00285 A_4' = 0.00866	A_{10}' =0.25488 A_9' =-1.46305 A_8' = 3.29048 A_7' =-3.43231 **87.60%** A_6' = 1.30583 A_5' = 0.20577	**61.24%** (=3rd degree parab./ 10th IDZ=1)

4. Conclusions

As a result of the discussed studies many original and important conclusions can be drawn. In Woźnica (2012) and Zboiński (2012) it was shown univocally that the polynomial TCs of odd degrees with the biggest possible number of their terms have the smallest values of QF. Therefore authors by this paper wanted to use such an approach in polynomial TCs of even degrees. It filled in the gap in the range of polynomial TCs' degrees from 5 to 11. Results of optimisation gave the conclusions that without any doubt the curves from Tab. 1 do not have a chance to be the optimum solution for railway TCs for their standard lengths. So, the main aim of research done for needs of this work – finding TCs' shapes better than standards TCs' shapes - is achieved by the authors.

Trying to clarify the fact that the curvatures, bends have a relatively small negative impact on vehicle dynamics authors could now make some hypotheses that can possibly explain the problem. It needs to investigate and, therefore, it can be both true and false.

a) It seems, that a beneficial effect on the body's response to curvature's bend can be a vehicle suspension system (the authors is aware that inappropriately selected suspension system can also worsen the response);
b) It is worth noting, that the shape of the curvature function does not map trajectory of the vehicle in plan. It is mapped by *y* coordinate, instead. So, bend in the curvature, as opposed to a bend in superelevation ramp does not cause direct bend in a trajectory. This is just the bend in course of the trajectory characteristic quantity, and not in the trajectory itself. This quantity has of course an important physical interpretation and can affect the dynamic behaviour, but it seems to be smaller than in the case of bends directly in the trajectory (track). Thus, bends in superelevation ramp should have greater importance. Confirmation of this reasoning are results of studies in Kuvfer (2000). Authors of this paper conclude there that, especially for high-speed rail, formation of shape of TC in the vertical direction must satisfy greater requirements than in the transverse direction. As a result, they propose a description of superelevation ramp function by curves of 2 degrees higher than the curvature function. Independently for conventional rail, the authors conclude on the basis of vehicle dynamics simulation, that polynomial and trigonometric TCs do not show the superiority over 3rd degree parabola. This is in certain accordance with the result obtained by authors of this proposal for the polynomials of 5th and 7th degree (Zboiński, 2012);
c) It is also worth noting, that the objective functions used by the authors so far refer to the lateral dynamics of the vehicle. The authors may incorporate in the analysis the quality function, which concerns the vertical dynamics of body mass centre. It may change both assessment of curves and results of the optimisation;
d) The authors also thinks, whether the relatively mild motion conditions adopted in the TC resulting from these ones specified in the regulations, should be changed for reasons of research for more severe, even if this would lead to unreal cases;

If none of the hypotheses appears to be true then the next idea is to modify QF calculation so that initial and end zones have bigger weights (importance) than the middle zone. Maybe bigger length of the middle zone causes that shape of the terminal zones has become less important.

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